On well-posedness in Gevrey classes of the Cauchy problem for hyperbolic operators of second order (examples)

Seiichiro Wakabayashi

1. Main theorem

$$
\begin{aligned}
&P(t,\tau,\xi) = \tau^2 - a(t,\xi) + b(t,\tau,\xi) + c(t) \quad ((t,\xi) \in [0,\infty) \times \mathbf{R}^n),\\
&a(t,\xi): \ \text{homo. poly. of } \xi \text{ of deg. } 2,\\
&\qquad a(t,\xi) \ge 0 \quad ((t,\xi) \in [0,\infty) \times \mathbf{R}^n,\\
&\qquad \text{real anal. fun. of } t\ .\\
&b(t,\tau,\xi) = b_0(t)\tau + b_1(t,\xi),\\
&b_1(t,\xi): \ \text{homo. poly. of } \xi \text{ of deg. } 1.
\end{aligned}
$$

Let $\kappa \ge 1$.

$$
\begin{aligned}
&f(t,x) \in \mathcal{E}^{(\kappa)}([0,\infty)\times\mathbf{R}^n) \ (\text{resp. } \mathcal{E}^{\{\kappa\}}([0,\infty)\times\mathbf{R}^n))\\
&\quad \overset{def}{\Longleftrightarrow}\\
&\forall T>0,\ \forall h>0,\ \exists C \equiv C_{T,h} > 0\\
&(\text{resp. } \forall T>0,\ \exists h>0,\ \exists C \equiv C_T > 0) \quad \text{s..t.}\\
&\quad |\partial_t^j \partial_x^\alpha f(t,x)| \le C h^{j+|\alpha|} (j+|\alpha|)!^\kappa\\
&\qquad \text{for } (t,x) \in [0,T]\times\mathbf{R}^n \text{ with } |x| \le T.
\end{aligned}
$$

Similarly we define $\mathcal{E}^{(\kappa)}([0,\infty))$, $\mathcal{E}^{(\kappa)}(\mathbf{R}^n)$, $\mathcal{E}^{\{\kappa\}}([0,\infty))$, $\mathcal{E}^{\{\kappa\}}(\mathbf{R}^n)$ and so on.

Assumption

(A) The coefficients of $b(t,\tau,\xi)$ and $c(t)$ belong to $\mathcal{E}^*([0,\infty))$.

Here $*$ denotes (κ) or $\{\kappa\}$.

We consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) \quad ((t,x)\in[0,\infty)\times\mathbf{R}^n), \\ u(0,x)=u_0(x), \quad (D_t u)(0,x)=u_1(x) \quad (x\in\mathbf{R}^n) \end{cases}$$

in $\mathcal{E}^*([0,\infty)\times\mathbf{R}^n)$, where

$$f(t,x)\in\mathcal{E}^*([0,\infty)\times\mathbf{R}^n), \quad u_0(x),u_1(x)\in\mathcal{E}^*(\mathbf{R}^n).$$

Note that

(i) if $g(x)\in\mathcal{D}^{(\kappa)}(\mathbf{R}^n)\equiv\mathcal{E}^{(\kappa)}(\mathbf{R}^n)\cap C_0^\infty(\mathbf{R}^n)$, then
$\forall A>0$, $\exists C_A>0$ s.t.

$$|\hat{g}(\xi)|\le C_A\exp[-A|\xi|^{1/\kappa}] \quad \text{for } \xi\in\mathbf{R}^n,$$

where

$$\hat{g}(\xi)=\int_{\mathbf{R}^n} g(x)e^{-ix\cdot\xi}\,dx : \quad \text{Fourier transform of } g,$$

and that

(ii) if $g(x)\in\mathcal{D}^{\{\kappa\}}(\mathbf{R}^n)\equiv\mathcal{E}^{\{\kappa\}}(\mathbf{R}^n)\cap C_0^\infty(\mathbf{R}^n)$, then
$\exists A>0$, $\exists C>0$ s.t.

$$|\hat{g}(\xi)|\le C\exp[-A|\xi|^{1/\kappa}] \quad \text{for } \xi\in\mathbf{R}^n.$$

Prop. If $1<\kappa\le 2$ (resp. $1<\kappa<2$), then (CP) is $\mathcal{E}^{(\kappa)}$ well-posed (resp. $\mathcal{E}^{\{\kappa\}}$ well-posed).

To state our main theorem, we need a set-valued function $\mathcal{R}(\xi)$: $S^{n-1}\ni\xi\mapsto\mathcal{R}(\xi)\in\mathcal{P}(\mathbf{C})$, which satisfies the following:

$$\forall T>0,\ \exists N_T\in\mathbf{Z}_+\ s.t.\ \#\{\lambda\in\mathcal{R}(\xi);\ \operatorname{Re}\lambda\le T\}\le N_T \quad \text{for } \xi\in S^{n-1}.$$

We note that $\mathcal{R}(\xi)$ is usually chosen as zeros of $a(\lambda,\xi)$ in λ with some modifications.

Main theorem Assume that $\kappa > 1$, and that
$0 \le \exists\nu < 1/\kappa$ s.t. "$\forall T > 0$ $\exists C_T > 0$ s.t.

$$\text{(G)} \qquad \Big(\min_{s\in\mathcal{R}(\xi)} |t-s|\Big)^{1/(1-\nu)} |b_1(t,\xi)| \le C_T\, a(t,\xi)^{(1-2\nu)/(2-2\nu)}$$
$$\text{for } (t,\xi) \in [0,T] \times S^{n-1}."$$

Then (CP) is $\mathcal{E}^*$ well-posed.

Remark (i) If we can take $\nu = 0$, (CP) is C^∞ well-posed. Here we may assume the coefficients of $b(t,\tau,\xi)$ and $c(t)$ belong to $C^\infty([0,\infty))$.
(ii) There is no difference in the results of Main theorem between for $\mathcal{E}^{(\kappa)}$ and for $\mathcal{E}^{\{\kappa\}}$ (see Prop.).
(iii) From Prop. it is enough to consider the case $\kappa \ge 2$. So we assume that $0 \le \nu < 1/2$ in (G).

2. Outline of the proof of Main theorem

Let $T > 0$ be fixed, and let $\Lambda(t,\xi)$ be a symbol in $\mathcal{E}^*$ satisfying, with some $C_T > 0$,

$$|\Lambda(t,\xi)| \le C_T\langle\xi\rangle^{1/\kappa} \quad \text{for } (t,\xi) \in [0,T] \times \mathbf{R}^n \text{ if } * = (\kappa),$$
$$|\Lambda(t,\xi)| \le C_T\langle\xi\rangle^{\nu} \quad \text{for } (t,\xi) \in [0,T] \times \mathbf{R}^n \text{ if } * = \{\kappa\},$$

where $\nu < 1/\kappa$ is fixed.

To consider (CP) in $\mathcal{E}^{(\kappa)}$
$\longleftrightarrow$
To consider, for $\forall A > 0$,

$$(\mathrm{CP})_A \quad \begin{cases} (\exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(t,D_x)]P(t,D_t,D_x) \\ \qquad \times \exp[-A\langle D_x\rangle^{1/\kappa} + \gamma\Lambda(t,D_x)])v(t,x) \\ \quad = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(t,D_x)]f(t,x), \\ \qquad\qquad\qquad\qquad (t,x) \in [0,T] \times \mathbf{R}^n, \\ v(0,x) = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(0,D_x)]u_0(x), \quad x \in \mathbf{R}^n, \\ (D_t v)(0,x) = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(0,D_x)]u_1(x), \quad x \in \mathbf{R}^n, \end{cases}$$

where we choose $\gamma > 0$ appropriately to obtain energy estimates and $f(t,x) \in C^\infty([0,T];\mathcal{D}^{(\kappa)}(\mathbf{R}^n)$, $u_k(x) \in \mathcal{D}^{(\kappa)}(\mathbf{R}^n)$ $(k = 1, 2)$.
$+\ \alpha$

If we can solve $(\mathrm{CP})_A$, then

$$u(t,x) = \exp[-A\langle D_x\rangle^{1/\kappa} + \gamma\Lambda(t, D_x)]v(t,x)$$

satisfies (CP).

To show that (CP) is $\mathcal{E}^{(\kappa)}$ well-posed we prove that (CP) has finite propagation property. In doing so, we replace $P(t,\tau,\xi)$ by

$$P_\varepsilon(t,\tau,\xi) = \tau^2 - a(t,\xi) - \varepsilon|\xi|^2 + b(t,\tau,\xi) + c(t),$$

where $\varepsilon \in (0, 1]$. Then we need to apply the same arguments as in

[W] W., On the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable, J. Math. Soc. Japan 62-1 (2010), 95–133.

So we first consider

$$(\mathrm{CP})_{\varepsilon,A}\quad \begin{cases} (\exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(t,D_x)]P_\varepsilon(t,D_t,D_x) \\ \qquad \times \exp[-A\langle D_x\rangle^{1/\kappa} + \gamma\Lambda(t,D_x)])v_\varepsilon(t,x) \\ \quad = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(t,D_x)]f(t,x), \\ \qquad\qquad\qquad\qquad (t,x) \in [0,T]\times\mathbf{R}^n, \\ v_\varepsilon(0,x) = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(0,D_x)]u_0(x), \quad x \in \mathbf{R}^n, \\ (D_t v_\varepsilon)(0,x) = \exp[A\langle D_x\rangle^{1/\kappa} - \gamma\Lambda(0,D_x)]u_1(x), \quad x \in \mathbf{R}^n, \end{cases}$$

where $A > 0$ is chosen so that

$$(1)\qquad \begin{cases} \exp[2A\langle D_x\rangle^{1/\kappa}]f(t,x) \in C^\infty([0,T];H^\infty(\mathbf{R}^n)), \\ \exp[2A\langle D_x\rangle^{1/\kappa}]u_k(x) \in H^\infty(\mathbf{R}^n) \quad (k = 0, 1). \end{cases}$$

Similarly, we have:

To consider (CP) in $\mathcal{E}^{\{\kappa\}}$

$\longleftrightarrow$

To choose $A > 0$ so as to satisfy (1), and to consider $(\mathrm{CP})_{\varepsilon,A}$ in the Sobolev spaces

$+ \alpha$

To prove Prop. we take

$$\Lambda(t,\xi) = t\langle\xi\rangle^{1/2}.$$

Let us give an outline of the proof of Main theorem. Put

$$\begin{aligned}\Lambda(t,\xi) =& t\langle\xi\rangle^{\nu} + \sum_{s\in\mathcal{R}(\xi/|\xi|)} \langle\xi\rangle^{\nu} \log\Big(\sqrt{(t-s)^2\langle\xi\rangle+1} + (t-s)\langle\xi\rangle^{1/2}\Big) \\ &+ \langle\xi\rangle^{\nu}\log\Big(\sqrt{t^2\langle\xi\rangle^{4/3}+1} + t\langle\xi\rangle^{2/3}\Big).\end{aligned}$$

We also put

$$\begin{aligned}&W(t,\xi) := \partial_t\Lambda(t,\xi) \\ &= \langle\xi\rangle^{\nu} + \sum_{s\in\mathcal{R}(\xi/|\xi|)} \langle\xi\rangle^{\nu+1/2}/\sqrt{(t-s)^2\langle\xi\rangle+1} + \langle\xi\rangle^{\nu+2/3}/\sqrt{t^2\langle\xi\rangle^{4/3}+1}.\end{aligned}$$

For $\varepsilon \in (0,1]$ and $\gamma > 0$ we define

$$\begin{aligned}\mathcal{E}_\varepsilon(t,\xi;w;\gamma) =& \exp[-\gamma\Lambda(t,\xi)] \\ &\times \{|\partial_t w(t,\xi)|^2 + (a(t,\xi)+\varepsilon|\xi|^2 + W(t,\xi)^2)|w(t,\xi)|^2\},\end{aligned}$$

and apply the arguments in §3 of [W].

$$\begin{aligned}&\partial_t\mathcal{E}_\varepsilon(t,\xi;w_\varepsilon;\gamma) \\ &\le [|\hat{g}(t,\xi)|^2/W(t,\xi) \\ &\quad - \{\gamma - 3 - (|c(t)| + 2\operatorname{Im} b_0(t))/W(t,\xi)\}W(t,\xi)|\partial_t w_\varepsilon(t,\xi)|^2 \\ &\quad - \{\gamma a(t,\xi)W(t,\xi)^2 + (\gamma-3)W(t,\xi)^4 - |b_1(t,\xi)|^2 \\ &\qquad - |c(t)|W(t,\xi) - \partial_t a(t,\xi)\cdot W(t,\xi)\}|w_\varepsilon(t,\xi)|^2/W(t,\xi)] \\ &\quad \times \exp[-\gamma\Lambda(t,\xi)],\end{aligned}$$

where

$$\hat{g}(t,\xi) = \exp[A\langle\xi\rangle^{1/\kappa} - \gamma\Lambda(t,\xi)]\hat{f}(t,\xi),$$

$$w_k(\xi) = \exp[A\langle\xi\rangle^{1/\kappa} - \gamma\Lambda(0,\xi)]\hat{u}_k(\xi) \quad (k=0,1),$$

$$\exp[3A\langle\xi\rangle^{1/\kappa}/4]\hat{g}(t,\xi) \in C^\infty([0,T];L^2(\mathbf{R}^n)), \tag{2}$$

$$\exp[3A\langle\xi\rangle^{1/\kappa}/4]w_k(\xi) \in L^2(\mathbf{R}^n) \quad (k=0,1), \tag{3}$$

$$\begin{cases} (\exp[-\gamma\Lambda(t,\xi)]P_\varepsilon(t,D_t,\xi)\exp[\gamma\Lambda(t,\xi)])w_\varepsilon(t,\xi) = \hat{g}(t,\xi), \\ w_\varepsilon(0,\xi) = w_0(\xi), \quad (\partial_t w_\varepsilon)(0,\xi) = w_1(\xi). \end{cases}$$

From the unique existence theorem of ordinary differential equations it follows that the solution $w_\varepsilon(t,\xi)$ uniquely exists.

Fix $T > 0$, and choose $\gamma > 0$ so that

$$\gamma - 3 - |c(t)| - 2\,\mathrm{Im}\, b_0(t) \geq 0 \quad (t \in [0,T]).$$

If we can show that

$$\partial_t a(t,\xi)\cdot W(t,\xi) \lesssim a(t,\xi)W(t,\xi)^2 + W(t,\xi)^4 \qquad ((t,\xi)\in[0,T]\times\mathbf{R}^n), \tag{4}$$

$$|b_1(t,\xi)|^2 \lesssim a(t,\xi)W(t,\xi)^2 + W(t,\xi)^4 \qquad ((t,\xi)\in[0,T]\times\mathbf{R}^n), \tag{5}$$

then, taking γ sufficiently large, we have

$$\partial_t \mathcal{E}_\varepsilon(t,\xi;w_\varepsilon;\gamma) \leq \exp[-\gamma\Lambda(t,\xi)]|\hat{g}(t,\xi)|^2/W(t,\xi).$$

Therefore, we have

$$\begin{aligned} &\mathcal{E}_\varepsilon(t,\xi;w_\varepsilon;\gamma) \\ &\leq \mathcal{E}_\varepsilon(0,\xi;w_\varepsilon;\gamma) + \int_0^t \exp[-\gamma\Lambda(s,\xi)]|\hat{g}(s,\xi)|^2/W(s,\xi)\,ds, \end{aligned} \tag{6}$$

which proves Main theorem after several steps(see [W]).

(4) was proved in [W] in the case $\nu = 0$, if necessary, modifying $\mathcal{R}(\xi)$. Let us prove that (5) holds. Fix $(t,\xi)\in[0,T]\times\mathbf{R}^n$.

(I) If $\exists s \in \mathcal{R}(\xi/|\xi|)$ s.t. $|t-s|\langle\xi\rangle^{1/2} \leq 1$, then $W(t,\xi) \geq \langle\xi\rangle^{\nu+1/2}/\sqrt{2}$, and, therefore, (5) holds.

(II) Assume that $|t-s|\langle\xi\rangle^{1/2} \geq 1$ for $\forall s \in \mathcal{R}(\xi/|\xi|)$. Then we have

$$W(t,\xi) \geq \frac{1}{\sqrt{2}}\langle\xi\rangle^\nu \Big(\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|\Big)^{-1}.$$

If we can show that, with some $X \in [0,1]$,

$$|b_1(t,\xi)| \lesssim \Big\{|\xi|^\nu \Big(\min_{s\in\mathcal{R}(\xi/|\xi|)}|t-s|\Big)^{-1}\sqrt{a(t,\xi)}\Big\}^X \tag{7}$$

$$\times \Big\{|\xi|^\nu \Big(\min_{s\in\mathcal{R}(\xi/|\xi|)} |t-s|\Big)^{-1}\Big\}^{2(1-X)},$$

then we can see that (5) holds. By homogenuity in ξ we have

$$1 = (\nu+1)X + 2(1-X)\nu \qquad \therefore X = \frac{1-2\nu}{1-\nu}.$$

The condition (G) is obtained by (7) with $X = \dfrac{1-2\nu}{1-\nu}$. Therefore, if (G) is satisfied, then (6) holds.
Fix $\gamma > 0$ which was chosen large so as to obtain (6). Choose ν' so that $\nu < \nu' < 1/\kappa$. Then we have

$$|\Lambda(t,\xi)| \le C_0\langle\xi\rangle^{\nu'} \quad \text{for } t\in[0,T].$$

Therefore, we have, with some $\widehat{C} > 0$,

$$\begin{aligned}&\exp[-\widehat{C}\langle\xi\rangle^{\nu'}]\{|w(t,\xi)|^2 + |\partial_t w(t,\xi)|^2\} \le \mathcal{E}_\varepsilon(t,\xi;w;\gamma)\\ &\le \exp[\widehat{C}\langle\xi\rangle^{\nu'}]\{|w(t,\xi)|^2 + |\partial_t w(t,\xi)|^2\} \quad (t\in[0,T]).\end{aligned}$$

(6) gives

$$\begin{aligned}(8)\quad &|w_\varepsilon(t,\xi)|^2 + |\partial_t w_\varepsilon(t,\xi)|^2\\ &\le \exp[2\widehat{C}\langle\xi\rangle^{\nu'}]\{|w_0(\xi)|^2 + |w_1(\xi)|^2\} + \int_0^t \exp[2\widehat{C}\langle\xi\rangle^{\nu'}]|\hat{g}(s,\xi)|^2\,ds.\end{aligned}$$

So by (2) and (3) we can see that $w_\varepsilon(t,\xi) \in C^\infty([0,T];L^2(\mathbf{R}^n))$, and that we can define the inverse Fourier transform of $w_\varepsilon(t,\xi)$ in ξ. Put

$$u_\varepsilon(t,x) = \exp[-A\langle D_x\rangle^{1/\kappa} + \gamma\Lambda(t,D_x)]\mathcal{F}_\xi^{-1}[w_\varepsilon(t,\xi)](x).$$

Then $u_\varepsilon(t,x)$ satisfies

$$\begin{cases} P_\varepsilon(t,D_t,D_x)u_\varepsilon(t,x) = f(t,x), \quad (t,x)\in[0,T]\times\mathbf{R}^n,\\ u_\varepsilon(0,x) = u_1(x), \quad (\partial_t u_\varepsilon)(0,x) = u_1(x), \quad x\in\mathbf{R}^n,\end{cases}$$

$$\begin{aligned}&|\hat{u}_\varepsilon(t,\xi)|^2 + |\partial_t\hat{u}_\varepsilon(t,\xi)|^2\\ &\le C\exp[-A\langle\xi\rangle^{1/\kappa}/2]\Big[\sum_{k=0}^{1} |\exp[2A\langle\xi\rangle^{1/\kappa}]\hat{u}_k(\xi)|^2\end{aligned}$$

$$+\int_0^t |\exp[2A\langle\xi\rangle^{1/\kappa}]\hat{f}(s,\xi)|^2\,ds\Big].$$

Since $P_\varepsilon(t,\tau,\xi)$ $(\varepsilon > 0)$ is strictly hyperbolic, we can estimate $\operatorname{supp} u_\varepsilon$. Noting that $u_\varepsilon(t,x) \to u(t,x)$ in $\mathcal{D}'((0,T)\times\mathbf{R}^n)$ as $\varepsilon \downarrow 0$, we can also estimate $\operatorname{supp} u$. Therefore, applying the same argument as in [W], we can construct a solution $u(t,x)$ of (CP) even if $u_0(x)$, $u_1(x)$ and $f(t,x)$ do not have compact supports with respect to x. It is obvious that $u(t,x) \in C^\infty([0,T];\mathcal{E}^*(\mathbf{R}^n))$. Let us prove that $u(t,x) \in \mathcal{E}^*([0,T]\times\mathbf{R}^n)$. By the equation we have

$$D_t^2\hat{u}(t,\xi) = \tilde{a}(t,\xi)\hat{u}(t,\xi) - b_0(t)D_t\hat{u}(t,\xi),$$

where $\tilde{a}(t,\xi) = a(t,\xi) - b_1(t,\xi) - c(t)$. We shall prove, by induction, that

$$(9)\quad |D_t^k\hat{u}(t,\xi)| \le C(u)A(u)^k k!^\kappa \sum_{l=0}^k \frac{\langle\xi\rangle^l}{l!^\kappa} B^l \exp[-A\langle\xi\rangle^{1/\kappa}/2] \quad (k \ge 0)$$

for $(t,\xi) \in [0,T]\times\mathbf{R}^n$, where $B > 0$ is a given constant. Let

$$|D_t^k\tilde{a}(t,\xi)| \le C(\tilde{a})A(\tilde{a})^k k!^\kappa \langle\xi\rangle^2,$$
$$|D_t^k b_0(t)| \le C(b_0)A(b_0)^k k!^\kappa$$

for $(t,\xi) \in [0,T]\times\mathbf{R}^n$. Then we choose $A(u) > 0$ so that

$$A(u) \ge A(b_0), \quad A(u) \ge \sqrt{2c_\kappa C(\tilde{a})}/B, \quad A(u) \ge 2c_\kappa C(b_0)/B$$

in (9). Here $c_\kappa > 0$ is a constant satisfying

$$\sum_{l=0}^k \binom{k}{l}^{1-\kappa} \le c_\kappa \quad (k \in \mathbf{Z}_+).$$

We may suppose that (9) is valid for $k = 0, 1$. Let $k \ge 0$. Then we have

$$\begin{aligned}
&|D_t^{k+2}\hat{u}(t,\xi)| \\
&\le \sum_{l=0}^k \binom{k}{l} |D_t^{k-l}\tilde{a}(t,\xi)\cdot D_t^l\hat{u}(t,\xi)| + \sum_{l=0}^k \binom{k}{l} |D_t^{k-l}b_0(t)\cdot D_t^{l+1}\hat{u}(t,\xi)| \\
&\le \sum_{l=0}^k \binom{k}{l} C(\tilde{a})A(\tilde{a})^{k-l}(k-l)!^\kappa\langle\xi\rangle^2\cdot C(u)A(u)^l l!^\kappa
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \sum_{\mu=0}^{l} \frac{\langle\xi\rangle^{\mu}}{\mu!^{\kappa}} B^{\mu} \exp[-A\langle\xi\rangle^{1/\kappa}/2] \\
&\quad + \sum_{l=0}^{k} \binom{k}{l} C(b_0) A(b_0)^{k-l} (k-l)!^{\kappa} \langle\xi\rangle \cdot C(u) A(u)^{l+1} (l+1)!^{\kappa} \\
&\quad \times \sum_{\mu=0}^{l+1} \frac{\langle\xi\rangle^{\mu}}{\mu!^{\kappa}} B^{\mu} \exp[-A\langle\xi\rangle^{1/\kappa}/2] \\
&\le C(u) A(u)^{k+2} (k+2)!^{\kappa} \Big[\frac{C(\tilde{a})}{A(u)^2} \sum_{l=0}^{k} \binom{k}{l}^{1-\kappa} \frac{1}{(k+2)^{\kappa}(k+1)^{\kappa}} \\
&\quad \times \Big(\frac{A(\tilde{a})}{A(u)}\Big)^{k-l} \sum_{\mu=0}^{l} \frac{\langle\xi\rangle^{\mu+2}}{\mu!^{\kappa}} B^{\mu} \\
&\quad + \frac{C(b_0)}{A(u)} \sum_{l=0}^{k} \binom{k}{l}^{1-\kappa} \Big(\frac{l+1}{(k+2)(k+1)}\Big)^{\kappa} \\
&\quad \times \Big(\frac{A(b_0)}{A(u)}\Big)^{k-l} \sum_{\mu=0}^{l+1} \frac{\langle\xi\rangle^{\mu+1}}{\mu!^{\kappa}} B^{\mu} \Big] \exp[-A\langle\xi\rangle^{1/\kappa}/2] \\
&\le C(u) A(u)^{k+2} (k+2)!^{\kappa} \\
&\quad \times \Big[\frac{C(\tilde{a})}{A(u)^2 (k+2)^{\kappa} (k+1)^{\kappa} B^2} \sum_{l=0}^{k} \binom{k}{l}^{1-\kappa} \Big(\frac{A(\tilde{a})}{A(u)}\Big)^{k-l} \\
&\quad \times \sum_{\mu=2}^{l+2} \frac{\langle\xi\rangle^{\mu}}{\mu!^{\kappa}} B^{\mu} \mu^{\kappa} (\mu-1)^{\kappa} \\
&\quad + \frac{C(b_0)}{A(u)(k+2)^{\kappa}(k+1)^{\kappa}} \sum_{l=0}^{k} \binom{k}{l}^{1-\kappa} \Big(\frac{A(b_0)}{A(u)}\Big)^{k-l} (l+1)^{\kappa} \\
&\quad \times \sum_{\mu=1}^{l+2} \frac{\langle\xi\rangle^{\mu}}{\mu!^{\kappa}} B^{\mu} \mu^{\kappa} \Big] \exp[-A\langle\xi\rangle^{1/\kappa}/2] \\
&\le C(u) A(u)^{k+2} (k+2)!^{\kappa} \sum_{\mu=0}^{k+2} \frac{\langle\xi\rangle^{\mu}}{\mu!^{\kappa}} B^{\mu} \\
&\quad \times \Big[\frac{c_\kappa C(\tilde{a})}{A(u)^2 B^2} + \frac{c_\kappa C(b_0)}{A(u) B} \Big] \exp[-A\langle\xi\rangle^{1/\kappa}/2]
\end{aligned}
$$

$$\leq C(u)A(u)^{k+2}(k+2)!^\kappa \sum_{\mu=0}^{k+2} \frac{\langle\xi\rangle^\mu}{\mu!^\kappa} B^\mu \exp[-A\langle\xi\rangle^{1/\kappa}/2]$$

for $(t,\xi) \in [0,T] \times \mathbf{R}^n$, which proves that (9) is valid.

Lemma Assume that $u(t,x) \in C^\infty([0,T];\mathcal{D}^*(\mathbf{R}^n))$ satisfies (9). Then we have

$$|\langle\xi\rangle^{n+1} D_t^k(\xi^\alpha \hat{u}(t,\xi))| \leq C(u)A(u)^k k!^\kappa \langle\xi\rangle^{n+1+|\alpha|} \exp[-A\langle\xi\rangle^{1/\kappa}/4]$$
$$\text{for } (t,\xi) \in [0,T] \times \mathbf{R}^n,\ k \in \mathbf{Z}_+ \text{ and } \alpha \in (\mathbf{Z}_+)^n.$$

Moreover, we have

$$|D_t^k D_x^\alpha u(t,x)| \leq C(u)' A(u)^k \Big(\frac{8\kappa}{A}\Big)^{\kappa|\alpha|} k!^\kappa |\alpha|!^\kappa$$
$$\text{for } (t,x) \in [0,T] \times \mathbf{R}^n,\ k \in \mathbf{Z}_+ \text{ and } \alpha \in (\mathbf{Z}_+)^n.$$

Remark When $* = (\kappa)$, we can take $A \to \infty$. Taking $B = (A/(4\kappa))^\kappa$, we have $A(u), (A/(4\kappa))^{-\kappa} \to 0$, *i.e.*, $u(t,x) \in \mathcal{E}^{(\kappa)}([0,T] \times \mathbf{R}^n)$.

Proof For $a_l \geq 0$ and $\kappa \geq 1$ it is obvious that

$$\Big(\sum_{l=1}^k a_l^\kappa\Big)^{1/\kappa} \leq \sum_{l=1}^k a_l \quad (k \in \mathbf{Z}_+).$$

So we have

$$\Big(\sum_{l=0}^k \frac{\langle\xi\rangle^l}{l!^\kappa} B^l\Big)^{1/\kappa} \leq \sum_{l=0}^k \frac{\langle\xi\rangle^{l/\kappa}}{l!} B^{l/\kappa} \leq \exp[B^{1/\kappa}\langle\xi\rangle^{1/\kappa}],$$
$$\sum_{l=0}^k \frac{\langle\xi\rangle^l}{l!^\kappa} B^l \leq \exp[\kappa B^{1/\kappa}\langle\xi\rangle^{1/\kappa}].$$

Therefore, (9) with $B = (A/(4\kappa))^\kappa$ yields

$$|\langle\xi\rangle^{n+1} D_t^k(\xi^\alpha \hat{u}(t,\xi))| \leq C(u)A(u)^k k!^\kappa \langle\xi\rangle^{n+1+|\alpha|} \exp[-A\langle\xi\rangle^{1/\kappa}/4].$$

Noting that

$$s^{k+l} \leq c^{-(k+l)\kappa}(k+l)!^\kappa \exp[\kappa c s^{1/\kappa}] \leq (c/2)^{-(k+l)\kappa} k!^\kappa l!^\kappa \exp[\kappa c s^{1/\kappa}]$$

for $c > 0$ and $k, l \in \mathbf{Z}_+$, we have

$$|\langle\xi\rangle^{n+1} D_t^k(\xi^\alpha \hat{u}(t,\xi))| \leq \widehat{C}(u)A(u)^k \Big(\frac{8\kappa}{A}\Big)^{\kappa|\alpha|} k!^\kappa |\alpha|!^\kappa,$$

which proves Lemma. □

3. Examples

Example 1 Let $n = 1$, $k, l \in \mathbf{Z}_+$, $a(t,\xi) = t^k\xi^2$ and $b_1(t,\xi) = t^l\xi$. We take

$$\mathcal{R}(\xi) = \{0\} \quad \text{for } \xi \in S^0 (= \{1, -1\}).$$

Taking $0 \le \nu < 1/\kappa$, the condition (G) is equivalent to

$$t^{l+1/(1-\nu)} \lesssim t^{k(1-2\nu)/(2-2\nu)} \quad (t \in [0,T]).$$

So we have

$$\begin{aligned} &(\mathrm{G}) \quad \Longleftrightarrow \quad l + \frac{1}{1-\nu} \ge \frac{k(1-2\nu)}{2-2\nu} \quad \Longleftrightarrow \\ &2l(1-\nu) + 2 \ge k(1-2\nu) \quad \Longleftrightarrow \quad (2k-2l)\nu \ge k - 2l - 2. \end{aligned}$$

Therefore, the condition (G) is equivalent to

$$\exists \nu \quad s.t. \qquad \kappa < \frac{1}{\nu} \le 1 + \frac{k+2}{(k-2l-2)_+}.$$

Here we consider $(k+2)/0 = \infty$. Indeed, if $k - 2l - 2 > 0$, then the above is obvious. When $k - 2l - 2 \le 0$,

$$\begin{aligned} &(2k-2l)\nu \ge k - 2l - 2 \quad \Longleftrightarrow \\ &k + 2 \ge (k-2l-2)(1/\nu - 1), \text{ which is valid.} \end{aligned}$$

Therefore, if

$$(\mathrm{I}) \qquad \kappa < 1 + \frac{k+2}{(k-2l-2)_+},$$

then (CP) is $\mathcal{E}^*$ well-posed.

Remark Ivrii proved that (I) is a necessary and sufficient condition for (CP) to be $\mathcal{E}^{\{\kappa\}}$ well-posed (when k is even)(?).

Example 2 Let $n = 2$, $k_j, l_j \in \mathbf{Z}_+$ $(j = 1, 2)$, $k_2 > 0$, and

$$a(t,\xi) = t^{k_1}(t^{k_2}\xi_1 - \xi_2)^2, \quad b_1(t,\xi) = t^{l_1}\xi_1 + t^{l_2}\xi_2.$$

We can write

$$b_1(t,\xi) = (t^{l_1}\xi_1 + t^{l_2+k_2})\xi_1 - t^{l_2}(t^{k_2}\xi_1 - \xi_2).$$

Put $\bar{l} = \min\{l_1, l_2 + k_2\}$. We take

$$\mathcal{R}(\xi) = \begin{cases} \{0, |\xi_2/\xi_1|^{1/k_2}\omega_1, \cdots, |\xi_2/\xi_1|^{1/k_2}\omega_{k_2}\} & \text{for } \xi \in S^1 \text{ with } \xi_1 \neq 0, \\ \{0\} & \text{for } \xi \in S^1 \text{ with } \xi_1 = 0, \end{cases}$$

where

$$\omega_l = \exp[i\{\arg(\xi_2/\xi_1) + 2(l-1)\pi\}/k_2] \quad (l = 1, 2, \cdots, k_2).$$

Taking account of Prop., we assume that $0 \leq \nu < 1/\kappa$ and $\nu \leq 1/2$. Noting that

$$\min_{s \in \mathcal{R}(\xi)} |t - s| \lesssim |t^{k_2} - \xi_2/\xi_1|^{1/k_2} \quad \text{if } \xi \in S^1 \text{ and } \xi_1 \neq 0,$$

we have

$$\text{(G)} \iff$$

$$\begin{cases} (1-2\nu)/(1-\nu) \leq 1, \\ t^{l_2 + 1/(1-\nu)} \lesssim t^{k_1(1-2\nu)/(2-2\nu)}, \\ \exists \lambda \in [0,1] \ s.t. \\ \quad t^{\bar{l} + (1-\lambda)/(1-\nu)} \lesssim t^{k_1(1-2\nu)/(2-2\nu)} |t^{k_2} - \xi_2/\xi_1|^{(1-2\nu)/(1-\nu) - \lambda/(k_2(1-\nu))} \\ \qquad\qquad \text{if } \xi_1 \neq 0, \\ t^{\bar{l} + 1/(1-\nu)} \lesssim t^{k_1(1-2\nu)/(2-2\nu)} : \quad \text{the condition for } \xi_1 \sim 0 \end{cases}$$

$$\iff$$

$$\begin{cases} (2k_1 - 2l_2)\nu \geq k_1 - 2l_2 - 2, \\ 0 \leq \lambda = k_2(1-2\nu) \leq 1, \\ (2k_1 + 4k_2 - 2\bar{l})\nu \geq k_1 + 2k_2 - 2\bar{l} - 2, \\ (2k_1 - 2\bar{l})\nu \geq k_1 - 2\bar{l} - 2 \end{cases}$$

$$\iff$$

$$\begin{cases} k_1 + 2 \geq (k_1 - 2l_2 - 2)(1/\nu - 1), \\ k_2 + 1 \geq (k_2 - 1)(1/\nu - 1), \\ k_1 + 2k_2 + 2 \geq (k_1 + 2k_2 - 2\bar{l} - 2)(1/\nu - 1), \\ k_1 + 2 \geq (k_1 - 2\bar{l} - 2)(1/\nu - 1) \end{cases}$$

$$\Longleftrightarrow$$

$$1+\min\Big\{\frac{k_1+2}{(k_1-2l_2-2)_+},\frac{k_2+1}{(k_2-1)_+},\frac{k_1+2k_2+2}{(k_1+2k_2-2\bar{l}-2)_+}\Big\}\geq\frac{1}{\nu}>\kappa.$$

Here we have used the inequality

$$\frac{k_1+2k_2+2}{(k_1+2k_2-2\bar{l}-2)_+}\leq\frac{k_1+2}{(k_1-2\bar{l}-2)_+}.$$

Therefore, if

$$\kappa<1+\min\Big\{\frac{k_1+2}{(k_1-2l_2-2)_+},\frac{k_2+1}{(k_2-1)_+},\frac{k_1+2k_2+2}{(k_1+2k_2-2\bar{l}-2)_+}\Big\},$$

then (CP) is $\mathcal{E}^*$ well-posed.