主部の係数が時間変数のみに依存する双曲型作用素について

2019.1.4∼1.6 松山キャンプ (山口大・理)
若林誠一郎 (筑波大学名誉教授)

1. 主結果

$m \in \mathbf{N}$, $P(t,x,\tau,\xi) = \tau^m + \sum_{j=1}^m \sum_{|\alpha|\le j} a_{j,\alpha}(t,x)\tau^{m-j}\xi^\alpha$, $x=(x_1,\cdots,x_n)\in\mathbf{R}^n$, $\xi=(\xi_1,\cdots,\xi_n)\in\mathbf{R}^n$, $a_{j,\alpha}(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$

Cauchy 問題

(CP) $$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty)\times\mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (\ 0\le j\le m-1) \end{cases}$$

Def. 1. Cauchy 問題 (CP) が C^∞ 適切
$\overset{def}{\Longleftrightarrow}$

(E) $\forall f\in C^\infty([0,\infty)\times\mathbf{R}^n)$, $\forall u_j\in C^\infty(\mathbf{R}^n)$ ($0\le j\le m-1$),
$\exists u\in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfying (CP).

(U) "$s>0$, $u\in C^\infty([0,\infty)\times\mathbf{R}^n)$, $D_t^j u(t,x)|_{t=0}=0$ ($0\le j\le m-1$) かつ
$P(t,x,D_t,D_x)u(t,x)=0$ for $t<s$"
$\Longrightarrow$
$u(t,x)=0$ for $t<s$.

$p(t,x,\tau,\xi)$: $P(t,x,\tau,\xi)$ の主部 (m 次斉次部分)
(Lax-Mizohata)
(CP) が C^∞ 適切
$\Longrightarrow$
$p(t,x,\tau,\xi)$ は各 $(t,x)\in[0,\infty)\times\mathbf{R}^n$ に対して hyperbolic w.r.t. $\vartheta=(1,0,\cdots,0)\in\mathbf{R}^{n+1}$, *i.e.*,
$p(t,x,\tau-i,\xi)\ne 0$ for $(t,x)\in[0,\infty)\times\mathbf{R}^n$, $(\tau,\xi)\in\mathbf{R}^{n+1}$
($\Leftrightarrow p(t,x,\tau,\xi)=\prod_{j=1}^m(\tau-\lambda_j(t,x,\xi))$, $\lambda_j(t,x,\xi)$ は実数値函数 ($\lambda_1(t,x,\xi)\le\cdots\le\lambda_m(t,x,\xi)$))

$h_j(t,x,\tau,\xi)$ ($\equiv h_j(t,x,\tau,\xi;p)$) ($0\le j\le m$) を

$$|p(t,x,\tau-i\gamma,\xi)|^2 = \sum_{j=0}^m \gamma^{2j} h_{m-j}(t,x,\tau,\xi) \quad (\ (\tau,\xi)\in\mathbf{R}^{n+1},\ \gamma\in\mathbf{R})$$

で定義.

仮定

(A-1) $a_{j,\alpha}(t,x) \equiv a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j$): $[0,\infty)$ で実解析的, *i.e.*,
(主部の係数が時間変数のみに依存しかつ $[0,\infty)$ で実解析的)
$\to \exists\Omega$: $[0,\infty)$ の複素近傍, $\exists\delta_0 > 0$ s.t.
$(-\delta_0,\infty) \subset \Omega$ かつ $a_{j,\alpha}(t)$ ($1 \le j \le m$, $|\alpha| = j$) は Ω で正則

(H) $p(t,\tau,\xi)$: hyperbolic w.r.t. ϑ for $t \in [-\delta_0,\infty)$

(A-2) $a_{j,\alpha}(t,x) \in C^\infty([0,\infty)\times \mathbf{R}^n)$ ($1 \le j \le m$, $|\alpha| = j-1$) かつ
$\forall R > 0$, $\exists C_R > 0$, $\exists A_R > 0$ s.t.

$$|\partial_t^k a_{j,\alpha}(t,x)| \le C_R A_R^k k! \quad \text{if } 1 \le j \le m,\ |\alpha| = j-1,\ k \in \mathbf{Z}_+,\ (t,x) \in [-\delta_0,R)\times\mathbf{R}^n,\ |x| \le R$$

(D) 2 重特性的, *i.e.*,

$$\partial_\tau^2 p(t,\tau,\xi) \ne 0 \quad \text{if } (t,\tau,\xi) \in [0,\infty)\times\mathbf{R}\times S^{n-1},\ p(t,\tau,\xi) = \partial_\tau p(t,\tau,\xi) = 0$$

ここで $S^{n-1} = \{\xi \in \mathbf{R}^n; |\xi| = 1\}$

$$\Gamma(p(t,\cdot),\vartheta) \stackrel{\leftarrow}{=} \text{“}\{(\tau,\xi) \in \mathbf{R}^{n+1}\setminus\{0\}; p(t,\tau,\xi) \ne 0\} \text{ の } \vartheta \text{ を含む連結成分”}$$

$(t_0,x^0) \in [0,\infty)\times\mathbf{R}^n$

$$\begin{aligned} K^\pm_{(t_0,x^0)} = \{&(t(s),x(s)) \in [0,\infty)\times\mathbf{R}^n;\ \pm s \ge 0 \text{ and } \{(t(s),x(s))\} \text{ is} \\ &\text{a Lipschitz cont. curve in } [0,\infty)\times\mathbf{R}^n \text{ satisfying} \\ &(d/ds)(t(s),x(s)) \in \Gamma(p(t,\cdot,\cdot),\vartheta)^* \ (\ a.e.\ s) \text{ and } (t(0),x(0)) = (t_0,x^0)\}, \end{aligned}$$

ここで $\Gamma^* = \{(t,x) \in \mathbf{R}^{n+1}; t\tau + x\cdot\xi \ge 0 \text{ for any } (\tau,\xi) \in \Gamma\}$.
$K^+_{(t_0,x^0)}$ は (t_0,x^0) の影響領域を記述,
$K^-_{(t_0,x^0)}$ は (t_0,x^0) の依存領域を記述する.

Levi 条件

$\mathcal{R}(\xi)$: $S^{n-1} \ni \xi \mapsto \mathcal{R}(\xi) \in \mathcal{P}(\mathbf{C})$: $\quad \forall T > 0$, $\exists N_T \in \mathbf{Z}_+$ s. t.

$$\#\{\lambda \in \mathcal{R}(\xi);\ \mathrm{Re}\ \lambda \in [0,T]\} \le N_T \quad \text{for } \xi \in S^{n-1}.$$

なる $\mathcal{R}(\xi)$ を 1 つ固定.

(L) $\forall T > 0$, $\forall x \in \mathbf{R}^n$, $\exists C > 0$ satisfying

$$\min\Big\{\min_{s\in\mathcal{R}(\xi)} |t-s|, 1\Big\} |sub\ \sigma(P)(t,x,\tau,\xi)| \le C h_{m-1}(t,\tau,\xi)^{1/2} \quad \text{for } (t,\tau,\xi) \in [0,T]\times\mathbf{R}\times S^{n-1},$$

ここで $\min_{s\in\mathcal{R}(\xi)} |t-s| = 1$ if $\mathcal{R}(\xi) = \emptyset$,

$$sub\ \sigma(P)(t,x,\tau,\xi) = P_{m-1}(t,x,\tau,\xi) + \frac{i}{2}\partial_t\partial_\tau p(t,\tau,\xi).$$

Thm 1 ([W9])**.** (A-1), (A-2), (H), (D), (L) を仮定. そのとき Cauchy 問題 (CP) は C^∞ 適切で, さらに

"$(t_0, x^0) \in (0,\infty) \times \mathbf{R}^n$ *and* $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$ *satisfies* (CP),
$u_j(x) = 0$ *near* $\{x \in \mathbf{R}^n; (0,x) \in K^-_{(t_0,x^0)}\}$ ($0 \le j \le m-1$) *and* $f = 0$ *near* $K^-_{(t_0,x^0)}$"
$\Longrightarrow$
$(t_0, x^0) \notin \operatorname{supp} u$

$$\mu_{j,k}(t,\xi) \stackrel{\leftarrow}{=} (\lambda_j(t,\xi) - \lambda_k(t,\xi))^2,$$
$$\prod_{1\le j<k\le m} (\tau - \mu_{j,k}(t,\xi)) = \tau^M + \sum_{l=1}^{M} (-1)^l D_l(t,\xi)\tau^{M-l}$$

によって, $\{D_\mu(t,\xi)\}_{1\le\mu\le M}$ を定義. ここで $M = \binom{m}{2}$.
注） $D_M(t,\xi)$ は $p(t,\tau,\xi) = 0$ in τ の判別式.

$D_0(t,\xi) \equiv 1$ とおき, 各 $\xi \in S^{n-1}$ に対して, $r(\xi) \in \mathbf{Z}_+$ を次で定義:

$$D_M(t,\xi) \equiv \cdots \equiv D_{M-r(\xi)+1}(t,\xi) \equiv 0 \text{ in } t, \quad D_{M-r(\xi)}(t,\xi) \not\equiv 0 \text{ in } t$$

さらに

$$\mathcal{R}_0(\xi) \stackrel{def}{=} \{(\mathrm{Re}\ \lambda)_+;\ \lambda \in \Omega,\ D_{M-r(\xi)}(\lambda,\xi) = 0\} \quad (\ \xi \in S^{n-1})$$

と定義する. そのとき, 必要なら Ω を修正して

$\forall T > 0$, $\exists N_T \in \mathbf{Z}_+$ s.t.

$$\#(\mathcal{R}_0(\xi) \cap [0,T]) \le N_T \quad \text{for } \xi \in S^{n-1}$$

が成立する (Lemma 2).

$S(\subset \mathbf{R}^n)$ が半代数的
$\stackrel{def}{\Longleftrightarrow}$
$\exists N \in \mathbf{N}$, $\exists r(j) \in \mathbf{N}$ ($1 \le j \le N$), $\exists A_{j,k} \subset \mathbf{R}^n$ ($1 \le j \le N$, $1 \le k \le r(j)$) s.t.
$A_{j,k}$ は実係数多項式の等式または不等式で定義される集合で, かつ

$$S = \bigcup_{j=1}^{N} \bigcap_{k=1}^{r(j)} A_{j,k}$$

$U\,(\subset \mathbf{R}^n)$: 半代数的, $h(t)$: U で定義された函数
$h(t)$ が U で半代数的
$\stackrel{def}{\Longleftrightarrow}$
グラフ $\{(t,h(t)) \in \mathbf{R}^2; t \in U\}$ が半代数的

Thm 2 ([W9])**.** (A-1), (A-2), (H), (D) を仮定. さらに $n \ge 3$ のとき, 各 $x \in \mathbf{R}^n$ に対して, $a_{j,\alpha}(t,x)$ ($1 \le j \le m$, $|\alpha| = j, j-1$) は $[0,\infty)$ で半代数的であると仮定する. そのとき (CP) が C^∞ 適切ならば

(L)$_0$ $\forall T > 0$, $\forall x \in \mathbf{R}^n$, $\exists C > 0$ *s.t.*

$$\min\Big\{\min_{s\in\mathcal{R}_0(\xi)} |t-s|, 1\Big\} |sub\ \sigma(P)(t,x,\tau,\xi)| \le C h_{m-1}(t,\tau,\xi)^{1/2}$$
$$for\ (t,\tau,\xi) \in [0,T] \times \mathbf{R} \times S^{n-1}$$

が成り立つ.

2. Thm 1(十分条件) の証明

2.1. 作用素の因数分解
2.2. 超局所エネルギー評価
2.3. 超局所エネルギー評価からエネルギー評価へ

2.1. $\mu \in \mathbf{R}$, I: $\mathbf{R}$ の区間

$a(t,x,\xi;\varepsilon) \in S^{\mu}_{\rho,\delta}(I \times T^*\mathbf{R}^n)$ uniformly in ε

$\overset{def}{\iff}$

$\exists C_{j,\alpha,\beta} > 0$ s.t. $C_{j,\alpha,\beta}$ は ε に indep. かつ

$$|D_t^j D_x^\beta \partial_\xi^\alpha a(t,x,\xi;\varepsilon)| \le C_{j,\alpha,\beta}\langle\xi\rangle^{\mu-\rho|\alpha|+\delta|\beta|} \quad \text{for } (t,x,\xi) \in I \times T^*\mathbf{R}^n,\ j \in \mathbf{Z}_+,\ \alpha,\beta \in (\mathbf{Z}_+)^n$$

$\kappa \in \mathbf{Z}_+$, $\kappa' \in \mathbf{Z}$

$a(t,x,\tau,\xi;\varepsilon) \in \mathcal{S}^{\kappa,\kappa'}_{1,0}$ uniformly in ε

$\overset{def}{\iff}$

$a(t,x,\tau,\xi;\varepsilon) = \sum_{j=0}^{\kappa} a_j(t,x,\xi;\varepsilon)\tau^j$,

$a_j(t,x,\xi;\varepsilon) \in S^{\kappa+\kappa'-j}_{1,0}(\mathbf{R} \times T^*\mathbf{R}^n)$ uniformly in ε: classical symbol (polyhomogeneous)

$$\mathcal{S}^{\kappa}_{1,0} \overset{\leftarrow}{\equiv} \mathcal{S}^{\kappa,0}_{1,0}, \qquad \mathcal{S}^{\kappa,-\infty}_{1,0} \overset{\leftarrow}{\equiv} \bigcap_{\kappa' \in \mathbf{Z}} \mathcal{S}^{\kappa,\kappa'}_{1,0}$$

仮定 (D) より

$\exists\delta_1 > 0$, $\exists N_0 \in \mathbf{N}$, $\exists\mathcal{C}_j, \mathcal{C}_{j,0}$: $\mathbf{R}^n\setminus\{0\}$ の開錐集合, $\exists r_j \in \mathbf{Z}_+$ ($1 \le j \le N_0$), $\exists\tilde{p}_{j,k}(t,\tau,\xi) \in \mathcal{S}^2_{1,0}$ ($1 \le j \le N_0$, $1 \le k \le r_j$), $\exists\tilde{p}_{j,r_j+1}(t,\tau,\xi) \in \mathcal{S}^{m-2r_j}_{1,0}$ ($1 \le j \le N_0$) s.t. $2r_j \le m$, $\tilde{p}_{j,k}(t,\tau,\xi)$: $|\xi| \ge 1/4$ で (τ,ξ) について正斉次, $3\delta_1 \le \delta_0$, $\bigcup_{l=0}^{N_0}\mathcal{C}_{l,0} \supset S^{n-1}$, $\mathcal{C}_{j,0} \Subset \mathcal{C}_j$ かつ

$$\begin{aligned}
&p(t,\tau,\xi) = \prod_{k=1}^{r_j+1}\tilde{p}_{j,k}(t,\tau,\xi)\\
&\qquad\qquad \text{for } (t,\tau,\xi) \in \mathcal{V}_j \equiv [-2\delta_1, 4\delta_1] \times \mathbf{R} \times \overline{\mathcal{C}}_j \text{ with } |\xi| \ge 1/4,\\
&\{\tau \in \mathbf{C};\ \tilde{p}_{j,k}(t,\tau,\xi) = 0\} \cap \{\tau \in \mathbf{C};\ \tilde{p}_{j,l}(t,\tau,\xi) = 0\} = \emptyset\\
&\qquad\qquad \text{if } k \ne l,\ (t,\xi) \in \mathcal{V}_j,\ |\xi| \ge 1/4,\\
&\partial_\tau\tilde{p}_{j,r_j+1}(t,\tau,\xi) \ne 0\\
&\quad \text{if } (t,\tau,\xi) \in \mathcal{V}_j,\ |\xi| \ge 1/4,\ \tilde{p}_{j,r_j+1}(t,\tau,\xi) = 0
\end{aligned}$$

($1 \le j \le N_0$).

$$\begin{aligned}
&p_{j,k}(t,\tau,\xi):\ \tilde{p}_{j,k}(t,\tau,\xi) \text{ の主シンボル},\\
&p_{j,k}(t,\tau,\xi) = (\tau - b_{j,k}(t,\xi))^2 - a_{j,k}(t,\xi) \quad (\ 1 \le k \le r_j),\\
&a_{j,k}(t,\xi) \ge 0, \quad b_{j,k}(t,\xi):\ \text{実数値函数},\\
&a_{j,k}(t,\xi) = 0 \quad \text{for some } (t,\xi) \in [-2\delta_1, 4\delta_1] \times \overline{\mathcal{C}}_j \text{ with } |\xi| \ge 1/4
\end{aligned}$$

◎ $P(t,x,\tau,\xi)$ を $t \ge 3\delta_1/2$ で修正して

$$P(t,x,\tau,\xi) = p(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p(t,\tau,\xi) \quad (t \ge 2\delta_1)$$

([W6] より $t \geq 2\delta_1$ のとき, $P(t,x,D_t,D_x)$ に対するエネルギー評価が得られる)

◎解の存在と有限伝播性を示すために, $f(t,x)$, $a_{j,\alpha}(t,x)$ ($1 \leq j \leq m$, $|\alpha| \leq j-1$) を $\mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ ($3/2 < 2$) の元 $f_\varepsilon(t,x)$, $a_{j,\alpha}(t,x;\varepsilon)$ ($0 < \varepsilon \leq 1$) で近似する ($P(t,x,\tau,\xi) \to P(t,x,\tau,\xi;\varepsilon)$). ここで, 開集合 $D\,(\subset \mathbf{R}^n)$ に対して

$$f(x) \in \mathcal{E}^{\{s\}}(D) \quad \overset{def}{\Longleftrightarrow} \quad \forall K\text{: } D \text{ のコンパクト集合}, \exists C>0, \exists A>0 \text{ s.t.}$$
$$|\partial^\alpha f(x)| \leq CA^{|\alpha|}(\alpha!)^s \quad \text{for } \alpha \in (\mathbf{Z}_+)^n \text{ and } x \in K$$

$f_\varepsilon \in \mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$, $\operatorname{supp} f_\varepsilon \subset \{t \geq 0\}$ のとき

$$(\mathrm{CP})_\varepsilon \qquad \begin{cases} P(t,x,D_t,D_x;\varepsilon)u_\varepsilon(t,x) = f_\varepsilon(t,x), \\ \operatorname{supp} u_\varepsilon \subset \{t \geq 0\} \end{cases}$$

は $\mathcal{E}^{\{3/2\}}(\mathbf{R}^{n+1})$ において, 一意解 u_ε をもつ. さらに

$$(t_0,x^0) \in (0,\infty)\times\mathbf{R}^n,\ f_\varepsilon = 0 \text{ near } K^-_{(t_0,x^0)} \quad \Longrightarrow \quad (t_0,x^0) \notin \operatorname{supp} u_\varepsilon$$

(*e.g.*, [W2])

$\varepsilon \downarrow 0$ のとき

$$\exists u \in C^\infty([0,2\delta_1];H^\infty(\mathbf{R}^n)) \text{ s.t. } u_\varepsilon \to u \text{ in } \mathcal{D}'((-\infty,2\delta_1)\times\mathbf{R}^n)$$

を示したい (ε について一様な $(\mathrm{CP})_\varepsilon$ に対するエネルギー評価を得たい!!)

簡単のために $\exists R>0$ s.t. $\operatorname{supp}_x a_{j,\alpha}(t,x;\varepsilon) \subset \{|x| \leq R\}$ ($1 \leq j \leq m$, $|\alpha| \leq j-1$) を仮定する.

<u>因数分解定理</u>(*e.g.*, [K])

$1 \leq j \leq N_0$, $(t,x,\tau,\xi) \in [-3\delta_1/2, 4\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times(\overline{\mathcal{C}}_j \setminus \{0\})$, $\varepsilon \in (0,1]$ に対して

$$\begin{aligned} P(t,x,\tau,\xi;\varepsilon) &= P_{j,1}(t,x,\tau,\xi;\varepsilon) \circ P_{j,2}(t,x,\tau,\xi;\varepsilon) \circ \cdots \circ P_{j,r_j+1}(t,x,\tau,\xi;\varepsilon) \\ &\qquad + R_j(t,x,\tau,\xi;\varepsilon), \end{aligned}$$
$$P_{j,k}(t,x,\tau,\xi;\varepsilon) \in \mathcal{S}^{m_{j,k}}_{1,0} \text{ uniformly in } \varepsilon\text{: 主シンボルが } p_{j,k}(t,\tau,\xi),$$
$$R_j(t,x,\tau,\xi;\varepsilon) \in \mathcal{S}^{m-1,-\infty}_{1,0} \text{ uniformly in } \varepsilon$$

ここで $m_{j,k} = \begin{cases} 2 & (\ 1 \leq k \leq r_j), \\ m-2r_j & (\ k = r_j+1), \end{cases}$

$$\sigma(a(t,x,D_t,D_x)b(t,x,D_t,D_x)) = a(t,x,\tau,\xi)\circ b(t,x,\tau,\xi)$$

<u>**Lemma**</u> **1.** $\exists c_{j,k,0}(t,x,\xi), c_{j,k,1}(t,\xi) \in S^{-1}_{1,0}(\mathbf{R}\times T^*\mathbf{R}^n)$ ($1 \leq j \leq N_0$, $1 \leq k \leq r_j$) *s.t.*

$$\begin{aligned} &sub\ \sigma(P_{j,k}(\cdot;\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \\ &= sub\ \sigma(P(\cdot;\varepsilon))(t,x,b_{j,k}(t,\xi),\xi) \Big/ \prod_{1\leq l\leq r_j+1,\, l\neq k} p_{j,l}(t,b_{j,k}(t,\xi),\xi) \\ &\quad + c_{j,k,0}(t,x,\xi)a_{j,k}(t,\xi) + c_{j,k,1}(t,\xi)\partial_t a_{j,k}(t,\xi) \end{aligned}$$

for $1 \leq j \leq N_0$, $1 \leq k \leq r_j$ *and* $(t,\tau,\xi) \in [0,3\delta_1]\times\mathbf{R}^n\times\mathcal{C}_j$ *with* $|\xi| \geq 1$

2.2.

[W4] で $m=2$ かつ低階の係数も x に依存しないときを扱った

$\to$ 低階の係数が x にも依存するとき, 重み函数を (t,ξ) のみの函数とするために, 次のように考えた.

$\mathcal{O}_0$: $t=0$ での一変数収束べき級数のつくる環 $\leftarrow$ 単項イデアル環

$$\begin{aligned}\mathfrak{M}_0 :=\{&(\beta_{j,\alpha}(t))_{j+|\alpha|=m-1} \in \mathcal{O}_0^{M'};\ \exists C>0,\ \exists\delta>0 \text{ s.t.}\\ &\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}\Big|\sum_{j+|\alpha|=m-1}\beta_{j,\alpha}(t)\tau^j\xi^\alpha\Big| \le Ch_{m-1}(t,\tau,\xi)^{1/2}\\ &\text{for } t\in[0,\delta],\ \tau\in\mathbf{R} \text{ and } \xi\in S^{n-1}\}\end{aligned}$$

ここで $M' := \binom{m+n-1}{m-1}$.

$\mathfrak{M}_0$ は $\mathcal{O}_0^{M'}$ の $\mathcal{O}_0$-部分加群で, 有限生成. 故に

$\exists\beta^\mu(t)\equiv(\beta^\mu_{j,\alpha}(t))_{j+|\alpha|=m-1}\in\mathfrak{M}_0$ ($1\le\mu\le r_0$) s.t.

$$\mathfrak{M}_0 = \Big\{\sum_{\mu=1}^{r_0} c_\mu(t)\beta^\mu(t);\ c_\mu(t)\in\mathcal{O}_0\ (\ 1\le\mu\le r_0)\Big\}$$

(L) より $\exists c_\mu(t,x)\in C^\infty([0,3\delta_1]\times\mathbf{R}^n)$ ($1\le\mu\le r_0$) s.t.

$$\begin{aligned}&\mathit{sub}\ \sigma(P)(t,x,\tau,\xi) = \sum_{\mu=1}^{r_0} c_\mu(t,x)\beta^\mu(t,\tau,\xi),\\ &\min\{\min_{s\in\mathcal{R}(\xi)}|t-s|,1\}|\beta^\mu(t,\tau,\xi)| \le Ch_{m-1}(t,\tau,\xi)^{1/2}\\ &\qquad\qquad\text{for } (t,\tau,\xi)\in[0,3\delta_1]\times\mathbf{R}\times S^{n-1}.\end{aligned}$$

ここで $\beta^\mu(t,\tau,\xi)=\sum_{j+|\alpha|=m-1}\beta^\mu_{j,\alpha}(t)\tau^j\xi^\alpha$.

$\mathcal{C}_{j,0}\Subset\mathcal{C}_{j,1}\Subset\mathcal{C}_{j,2}\Subset\mathcal{C}_{j,3}\Subset\mathcal{C}_{j,4}\Subset\mathcal{C}_j$: 開錐集合の列

$1\le j\le N_0$ なる j を 1 つ固定して. 添え字の j を省略する

(*i.e.*, $P_{j,k}\to P_k$, $\mathcal{C}_j\to\mathcal{C}$, $r_j\to r$, $\cdots$).

$\Psi(\xi)$, $\varphi(\xi)\in S^0_{1,0}$:

$$\Psi(\xi) = \begin{cases} 1 & \text{if } \xi\in\mathcal{C}_1 \text{ and } |\xi|\ge 1,\\ 0 & \text{if } \xi\notin\mathcal{C}_2 \text{ or } |\xi|\le 1/2,\end{cases}$$

$$\varphi(\xi) = \begin{cases} 0 & \text{if } \xi\in\mathcal{C}_0 \text{ or } |\xi|\le 1/4,\\ 1 & \text{if } \xi\notin\mathcal{C}_1 \text{ and } |\xi|\ge 1/2\end{cases}$$

$\gamma\ge 1$, $\Psi_\gamma(\xi):=\Psi(\xi/\gamma)$, $\varphi_\gamma(\xi)=\cdots$ とおく.

<u>超局所化</u>

$$\begin{aligned}\Psi_\gamma(D_x)P(t,x,D_t,D_x;\varepsilon)u_\varepsilon &= \Psi_\gamma(D_x)f_\varepsilon,\\ P(\cdot;\varepsilon)(\Psi_\gamma(D_x)u_\varepsilon) &= \Psi_\gamma f_\varepsilon + [P,\Psi_\gamma]u_\varepsilon\end{aligned}$$

ここで $[P,\Psi_\gamma]=P\Psi_\gamma-\Psi_\gamma P$: 交換子

交換子の処理([KW2])

$B\geq 1$, $\Lambda(\xi):=\varphi(\xi)\log(1+\langle\xi\rangle)$ に対して

$$P_{B\Lambda}(t,x,\tau,\xi;\varepsilon):=e^{-B\Lambda(\xi)}\circ P(t,x,\tau,\xi;\varepsilon)\circ e^{B\Lambda(\xi)}$$

とおいて

$$P_{B\Lambda}(t,x,D_t,D_x;\varepsilon)(e^{-B\Lambda}\Psi_\gamma u_\varepsilon)=e^{-B\Lambda}\Psi_\gamma f_\varepsilon+e^{-B\Lambda}[P,\Psi_\gamma]u_\varepsilon\overset{\rightarrow}{=}g_\varepsilon(t,x;B)$$

"$\sigma([P,\Psi_\gamma])$ の (essential) support"$\cap\{|\xi|\geq\gamma\}\subset(\mathcal{C}_2\setminus\mathcal{C}_1)\cap\{|\xi|\geq 1\}\subset\{\varphi(\xi)=1\}$ より

$$e^{-B\Lambda}[P,\Psi_\gamma]\approx\langle D_x\rangle^{-B}[P,\Psi_\gamma]$$

・ $(P_{r+1})_{B\Lambda}(t,x,\tau,\xi;\varepsilon)$ は $(t,x,\tau,\xi)\in[-3\delta_1/2,4\delta_1]\times\mathbf{R}^n\times\mathbf{R}\times(\overline{\mathcal{C}}\setminus\{0\})$, $\varepsilon\in(0,1]$ で strictly hyp. より, 超局所エネルギー評価は既知.

◎ $(P_k)_{B\Lambda}(t,x,D_t,D_x;\varepsilon)\equiv(D_t-b_k(t,D_x))^2-a_k(t,D_x)+\sum_{l=0}^{1}P_{k,l}(t,x,D_t,D_x;\varepsilon,B)$ ($1\leq k\leq r$) に対する超局所エネルギー評価

← 擬微分作用素のカルキュラスが必要

← Hörmander metric と weight を定義して, エネルギー形式を定義

$$\kappa_k(\xi)\overset{\leftarrow}{=}\int_0^{3\delta_1}a_k(t,\xi)\,dt\quad\text{for }\xi\in\overline{\mathcal{C}}$$

とおく.

Lemma 2. $\exists m_0\in\mathbf{N}$, $\exists C>0$ *s.t.* $\forall\xi\in\overline{\mathcal{C}}\setminus\{0\}$, $\exists m_k(\xi)\in\mathbf{Z}_+$, $\exists a_{k,\mu}(\xi)\in\mathbf{R}$ ($1\leq\mu\leq m_k(\xi)$) *satisfying* $m_k(\xi)\leq m_0$ *and*

$$\begin{aligned}&C^{-1}\kappa_k(\xi)|t^{m_k(\xi)}+a_{k,1}(\xi)t^{m_k(\xi)-1}+\cdots+a_{k,m_k(\xi)}(\xi)|\leq a_k(t,\xi)\leq C\kappa_k(\xi),\\&|\partial_t a_k(t,\xi)|\leq C\kappa_k(\xi)\\&\quad\textit{for } t\in[0,3\delta_1]\end{aligned}$$

注）広中の特異点解消定理と Weierstrass の予備定理を用いて示せる.

$$\widetilde{\Psi}(\xi)\in S^0_{1,0}:\quad\widetilde{\Psi}(\xi)=\begin{cases}1 & \text{if }\xi\in\mathcal{C}_4\text{ and }|\xi|\geq 1/2,\\0 & \text{if }\xi\notin\mathcal{C}\text{ or }|\xi|\leq 1/4,\end{cases}\quad 0\leq\widetilde{\Psi}(\xi)\leq 1,$$

$$[\![\,\xi\,]\!]_k:=\sqrt{\kappa_k(\xi)\widetilde{\Psi}(\xi)+1}\quad\text{for }\xi\in\mathbf{R}^n$$

とする.

Lemma 3. $s\in\mathbf{R}$, $\alpha\in(\mathbf{Z}_+)^n$ に対して $\exists C_{s,\alpha}>0$ *satisfying*

$$|\partial^\alpha[\![\,\xi\,]\!]_k^s|\leq C_{s,\alpha}[\![\,\xi\,]\!]_k^{s-|\alpha|}$$

Lemma 2 で $m_0 \geq 2$ と仮定してよい.

$$
\begin{aligned}
&\rho_0 := 2/(m_0+2),\\
&w_k(t,\xi) := a_k(t,\xi)\widetilde{\Psi}(\xi) + [\![\,\xi\,]\!]_k^{2\rho_0},\\
&W_{k,0}(t,\xi) := [\![\,\xi\,]\!]_k^{2\rho_0}\, w_k(t,\xi)^{-1/2} + 1,\\
&W_{k,1}(t,\xi)\\
&:= \Big(\sum_{\mu=1}^{r_0} \widetilde{\Psi}(\xi)^2 |\beta^\mu(t, b_k(t,\xi),\xi)|^2 |\xi|^{-2m+4} + [\![\,\xi\,]\!]_k^{2\rho_0}\Big)^{1/2} w_k(t,\xi)^{-1/2} + 1,\\
&W_{k,2,1}(t,\xi) := (\widetilde{\Psi}(\xi)^4 |\partial_t a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{1/2}/w_k(t,\xi),\\
&W_{k,2,2}(t,\xi)\\
&\approx (\widetilde{\Psi}(\xi)^4 |\partial_t \nabla_\xi a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{1/2} (\widetilde{\Psi}(\xi)^4 |\nabla_\xi a_k(t,\xi)|^2 + [\![\,\xi\,]\!]_k^{2\rho_0})^{-1/2}\\
&\qquad\qquad\qquad (\text{ mollifier で右辺を修正}),\\
&W_{k,2}(t,\xi) := \sum_{l=1}^{2} W_{k,2,l}(t,\xi), \quad W_k(t,\xi) := \sum_{l=0}^{2} W_{k,l}(t,\xi)
\end{aligned}
$$

for $(t,\xi) \in [0, 3\delta_1] \times \mathbf{R}^n$

$\mathbf{R}^{2n}$ 上の Riemannian metric $g_{k,\rho}$ を

$$g_{k,\rho\,(x,\xi)}(y,\eta) = |y|^2 + [\![\,\xi\,]\!]_k^{-2\rho}\, |\eta|^2$$

で定義する. ここで $0 < \rho \leq \rho_0$.

Lemma 4. (i) $\exists c > 0$, $\exists C > 0$ *s.t.*

$$
\begin{aligned}
&g_{k,\rho\,(x+y,\xi+\eta)}(X) \leq C g_{k,\rho\,(x,\xi)}(X) \quad (\text{ slowly varying}),\\
&C^{-1}\,[\![\,\xi\,]\!]_k \leq [\![\,\xi+\eta\,]\!]_k \leq C\,[\![\,\xi\,]\!]_k \quad (\ g_{k,\rho} \text{ cont.})\\
&\qquad \textit{if } (x,\xi),\ (y,\eta),\ X \in \mathbf{R}^{2n},\ g_{k,\rho\,(x,\xi)}(y,\eta) \leq c
\end{aligned}
$$

(ii)

$$g^\sigma_{k,\rho\,(x,\xi)}(y,\eta) \Big(\equiv \sup_X |\sigma((y,\eta),X)|^2 / g_{k,\rho\,(x,\xi)}(X)\Big) = [\![\,\xi\,]\!]_k^{2\rho}\, |y|^2 + |\eta|^2$$

(iii) $\exists C > 0$ *s.t.*

$$
\begin{aligned}
&g_{k,\rho\,(x,\xi)}(X) \leq C g_{k,\rho\,(x+y,\xi+\eta)}(X)(1 + g^\sigma_{k,\rho\,(x,\xi)}(y,\eta))^\rho \quad (\ \sigma \text{ temperate}),\\
&[\![\,\xi+\eta\,]\!]_k \leq C\,[\![\,\xi\,]\!]_k (1 + g^\sigma_{k,\rho\,(x,\xi)}(y,\eta))^{1/2} \quad (\ \sigma,\ g_{k,\rho} \text{ temperate})
\end{aligned}
$$

($g_{k,\rho}$ は Hörmander metric で, $[\![\,\xi\,]\!]_k$ は Hörmander weight)

Lemma 5. $\exists C_\alpha > 0$, $\exists C_{s,\alpha} > 0$ ($s \in \mathbf{R}$, $\alpha \in (\mathbf{Z}_+)^n$) *s.t.*

$$
\begin{aligned}
&|\partial_t w_k(t,\xi)\widetilde{\Psi}(\xi)| \leq W_{k,2}(t,\xi) w_k(t,\xi),\\
&|\partial_\xi^\alpha w_k(t,\xi)^s| \leq C_{s,\alpha} w_k(t,\xi)^s\, [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0} \quad (\ s \in \mathbf{R}),\\
&|\partial_\xi^\alpha W_{k,\mu}(t,\xi)| \leq C_\alpha W_{k,\mu}(t,\xi)\, [\![\,\xi\,]\!]_k^{-|\alpha|\rho_0} \quad (\ 0 \leq \mu \leq 2)\\
&\qquad \textit{for } \alpha \in (\mathbf{Z}_+)^n,\ (t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n
\end{aligned}
$$

さらに $W_{k,1}(t,\xi)$ は *uniformly* σ, g_{k,ρ_0} *temperate in* $t \in [0, 3\delta_1]$.

$$\Phi_k(t,\xi) \overset{def}{=} \int_0^t W_k(s,\xi)\,ds \quad \text{for } (t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$$

Lemma 6. $\exists C_\alpha > 0$ ($\alpha \in (\mathbf{Z}_+)^n$) *s.t.*

$$|\partial_\xi^\alpha \Phi_k(t,\xi)| \le C_\alpha (1 + \log [\![\xi]\!]_k) [\![\xi]\!]_k^{-|\alpha|\rho_0} \quad \textit{for } (t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n,\ \alpha \in (\mathbf{Z}_+)^n$$

注） Lemma 2 と [CIO] の手法を用いて示せる．また

$$\int_0^{3\delta_1} W_{k,1}(s,\xi)\,ds \le C(1 + \log [\![\xi]\!]_k)$$

を示すのに, 仮定 (L) を用いる.

$A > 0$, $\gamma \ge 1$, $l \in \mathbf{R}$ に対して

$$K_k(t,\xi;A,\gamma,l) := e^{-2\gamma t}\widetilde{K}_k(t,\xi;A,\gamma,l),$$
$$\widetilde{K}_k(t,\xi;A,\gamma,l) := \exp[-A\Phi_k(t,\xi) - 2t\log\langle\xi\rangle_\gamma + 2l\log\langle\xi\rangle_\gamma].$$

とおく．ここで $\langle\xi\rangle_\gamma = (\gamma^2 + |\xi|^2)^{1/2}$

$0 < \rho < \rho_0$ なる ρ を 1 つ固定.

Lemma 7. $\exists C_\alpha(A,l) > 0$ ($\alpha \in (\mathbf{Z}_+)^n$) *s.t.*

$$|\partial_\xi^\alpha \widetilde{K}_k(t,\xi;A,\gamma,l)| \le C_\alpha(A,l)\widetilde{K}_k(t,\xi;A,\gamma,l) [\![\xi]\!]_k^{-|\alpha|\rho} \quad \textit{for } \alpha \in (\mathbf{Z}_+)^n,\ (t,\xi) \in [0,3\delta_1] \times \mathbf{R}^n$$

さらに $\widetilde{K}_k(t,\xi;A,\gamma,l)$ は *uniformly* σ, $g_{k,\rho}$ *temperate in* $t \in [0,3\delta_1]$ (Hörmander weight)

$$\psi(\xi) \in S^0_{1,0}: \quad \psi(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{C}_3 \text{ and } |\xi| \ge 1/2, \\ 0 & \text{if } \xi \notin \mathcal{C}_4 \text{ or } |\xi| \le 1/4, \end{cases} \quad \psi_\gamma(\xi) := \psi(\xi/\gamma)$$

エネルギー形式

$$\begin{aligned} \mathcal{E}_k(t;w,A,\gamma,l) =& ((D_t - b_k(t,D_x))\psi_\gamma(D_x)w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &+ ((w_k(t,D_x) + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &\text{for } w(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n_x)) \text{ with } w|_{t\le 0} = 0 \text{ and } t \in [0,3\delta_1] \end{aligned}$$

ここで $K_k = K_k(t,D_x;A,\gamma,l)$. そのとき

$$\begin{aligned} &D_t\mathcal{E}_k(t;w,A,\gamma,l) \\ &= 2i\mathrm{Im}\ (\mathrm{Op}((\tau - b_k(t,\xi))^2)\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &+ 2i\mathrm{Re}\ (\mathrm{Op}(\partial_t b_k(t,\xi))\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &+ i((D_t - b_k)\psi_\gamma w, (AW_k(t,D_x) + 2(\gamma + \log\langle D_x\rangle_\gamma))K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &- 2i\mathrm{Im}\ ((w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &- i(\mathrm{Op}(\partial_t a_k(t,\xi))\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \\ &+ i((w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, (AW_k + 2(\gamma + \log\langle D_x\rangle_\gamma))K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)} \end{aligned}$$

ここで $\mathrm{Op}(s(t,x,\tau,\xi)) := s(t,x,D_t,D_x)$. また $\mathrm{Im}\ (b_k v, K_k v)_{L^2(\mathbf{R}^n_x)} = 0$ を用いた.

$$\begin{aligned}&\{(\tau - b_k(t,\xi))^2 + i\partial_t b_k(t,\xi)\}\psi_\gamma(\xi)\\&= [a_k(t,\xi) + (P_k)_{B\Lambda}(t,x,\tau,\xi;\varepsilon) - sub\ \sigma(P_k)(t,x,\tau,\xi;\varepsilon) - q^0_k(t,x,\tau,\xi;\varepsilon,B)]\psi_\gamma(\xi),\\&q^0_k(t,x,\tau,\xi;\varepsilon,B)/\log(1+\langle\xi\rangle) \in \mathcal{S}^{1,-1}_{1,0} \quad \text{uniformly in } \varepsilon\end{aligned}$$

故に

$$\begin{aligned}\partial_t\mathcal{E}_k(t;w,A,\gamma,l) &\le \|K^{1/2}_k(P_k)_{B\Lambda}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|K^{1/2}_k W^{-1/2}_{k,1} sub\ \sigma(P_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\\&+ \|K^{1/2}_k q^0_k\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|K^{1/2}_k W^{-1/2}_k [\![D_x]\!]^{2\rho_0}_k \psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|K^{1/2}_k(\log\langle D_x\rangle_\gamma)^{3/2}\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\\&+ \|K^{1/2}_k W^{-1/2}_k w^{-1/2}_k \mathrm{Op}(\partial_t a_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}/2\\&- ((D_t - b_k)\psi_\gamma w, ((A-1)W_k + 2\gamma + \log\langle D_x\rangle_\gamma - W_{k,1} - 2)K_k(D_t - b_k)\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\\&- (((A-1/2)W_k + 2\gamma + 2\log\langle D_x\rangle_\gamma)(w_k + (\log\langle D_x\rangle_\gamma)^2)\psi_\gamma w, K_k\psi_\gamma w)_{L^2(\mathbf{R}^n_x)}\end{aligned}$$

Lemma 8. $\kappa \in \mathbf{R}$, $q(t,x,\xi;\varepsilon,B) \in S^\kappa_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$ *uniformly in* ε に対して

$$\begin{aligned}&((K^{1/2}_k W^{-1/2}_{k,1}) \circ q(t,x,\xi;\varepsilon,B) \circ (K^{-1/2}_k W^{1/2}_{k,1}) - q(t,x,\xi;\varepsilon,B))\,[\![\xi]\!]^\rho_k\\&\in S^\kappa_{0,0}([0,3\delta_1]\times T^*\mathbf{R}^n) \quad \textit{uniformly in } \gamma \textit{ and } \varepsilon\end{aligned}$$

注）ここのみに, 擬微分作用素のカルキュラスが必要.

$$\begin{aligned}sub\ \sigma(P_k)(t,x,\tau,\xi;\varepsilon) &= q^1_{k,0}(t,x,\xi;\varepsilon)(\tau - b_k(t,\xi)) + \sum_{\mu=1}^{r_0}\tilde{c}_\mu(t,x)d_k(t,\xi)\beta^\mu(t,b_k(t,\xi),\xi)/|\xi|^{m-2}\\&+ c_{k,0}(t,x,\xi)a_k(t,\xi) + c_{k,1}(t,\xi)\partial_t a_k(t,\xi) \quad \text{for } (t,x,\xi)\in[0,3\delta_1]\times\mathbf{R}^n\times\overline{\mathcal{C}} \text{ with } |\xi|\ge 1\end{aligned}$$

ここで $d_k(t,\xi) \in S^0_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$, $q^1_{k,0}(t,x,\xi;\varepsilon) \in S^0_{1,0}(\mathbf{R}\times T^*\mathbf{R}^n)$ uniformly in ε, $\tilde{c}_\mu(t,x) \in C^\infty([0,3\delta_1]\times\mathbf{R}^n)$, $c_{k,l}(t,x,\xi) \in S^{-1}_{1,0}([0,3\delta_1]\times T^*\mathbf{R}^n)$ ($l = 0,1$).
これより $\exists C_0 > 0$, $\exists C(A,l) > 0$ s.t.

$$\begin{aligned}\|K^{1/2}_k W^{-1/2}_{k,1} sub\ \sigma(P_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} &\le r_0 C_0\|K^{1/2}_k W^{1/2}_{k,1} w^{1/2}_k\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)}\\&+ C(A,l)(\|K^{1/2}_k(D_t - b_k)\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)} + \|K^{1/2}_k w^{1/2}_k\psi_\gamma w\|^2_{L^2(\mathbf{R}^n_x)})\\&\text{for } t\in[0,3\delta_1]\end{aligned}$$

故に $\exists\widehat{C}_0 > 0$, $\exists\widehat{C}(A,l) \ge 1$ s.t.

$$\begin{aligned}&\mathcal{E}_k(t;w,A,\gamma,l) \le \int_0^t \|K_k(s,D_x)^{1/2}(P_k)_{B\Lambda}\psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\&\qquad \text{if } t\in[0,3\delta_1],\ A\ge\widehat{C}_0,\ \gamma\ge\widehat{C}(A,l)\end{aligned}$$

Lemma 9. $\exists\gamma_0(B)\ge 1$, $\exists\nu_1 > 0$, $\exists C_l(B) > 0$, $\exists C_{l,N}(B) > 0$ *s.t.*

$$\begin{aligned}&\sum_{\mu=0}^m\int_0^t \|e^{-\gamma s}D^\mu_t\langle D_x\rangle^{l-\mu}_\gamma e^{-B\Lambda}\Psi_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\&\le C_l(B)\int_0^t \|e^{-\gamma s}\langle D_x\rangle^{l+\nu_1}_\gamma g_\varepsilon(s,x;B)\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\&\quad + C_{l,N}(B)\sum_{\mu=0}^{m-1}\int_0^t \|e^{-\gamma s}D^\mu_t\langle D_x\rangle^{l-N-\mu}_\gamma w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\,ds\\&\textit{if } B\ge 1,\ l\in\mathbf{R},\ N\in\mathbf{N},\ t\in[0,3\delta_1],\ \gamma\ge\gamma_0(B)\end{aligned}$$

2.3.

$$\Theta(t) \in C^\infty(\mathbf{R}) : \quad \Theta(t) = \begin{cases} 1 & \text{if } t \le 3/2, \\ 0 & \text{if } t \ge 2, \end{cases} \quad \Theta_\gamma(\xi) := \Theta(|\xi|/\gamma)$$

として, j について足し合わせて

$$\begin{aligned}
&\sum_{\mu=0}^{m} \int_0^t \|e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{l-\mu} (1 - \Theta_\gamma(D_x)) w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds \\
&\le C_l(B) \Big\{ \int_0^t \|e^{-\gamma s} \langle D_x \rangle_\gamma^{l+\nu_1} P(t,x,D_t,D_x;\varepsilon) w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds \\
&\quad + \sum_{\mu=0}^{m-1} \int_0^t \|e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{m+l+\nu_1-\mu-1} \Theta_\gamma(D_x) w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds \\
&\quad + \gamma^{-1} \sum_{\mu=0}^{m-1} \int_0^t \|e^{-\gamma s} D_t^\mu \langle D_x \rangle_\gamma^{l-\mu} w|_{t=s}\|^2_{L^2(\mathbf{R}^n_x)}\, ds \Big\} \\
&\text{if } B \ge \nu_1 + 1,\ l \in \mathbf{R},\ t \in [0, 3\delta_1],\ \gamma \ge \gamma_0(B)
\end{aligned}$$

$(t,x,\tau,\xi) \in [-2\delta_1, 4\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, $|\xi| \le 2\gamma$ のとき
$\exists c_0 > 0$ s.t.

$$|P(t,x,\tau - i\gamma, \xi; \varepsilon)| \ge c_0 \langle (\tau,\xi) \rangle_\gamma^m$$

$\to \quad \exists C_{j,k,\alpha,\beta} > 0$ s.t.

$$\begin{aligned}
&|D_t^k D_x^\beta \partial_\tau^j \partial_\xi^\alpha P(t,x,\tau - i\gamma, \xi; \varepsilon)^{-1}| \le C_{j,k,\alpha,\beta} \langle (\tau,\xi) \rangle_\gamma^{-m-j-|\alpha|} \\
&\quad \text{for } (t,x,\tau,\xi) \in [-2\delta_1, 4\delta_1] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \text{ with } |\xi| \le 2\gamma
\end{aligned}$$

$$\chi_0(t) \in C^\infty(\mathbf{R}) : \quad \chi_0(t) = \begin{cases} 1 & \text{if } -\delta_1 \le t \le 3\delta_1, \\ 0 & \text{if } t \le -2\delta_1 \text{ or } t \ge 4\delta_1 \end{cases}$$

とし,

$$\begin{aligned}
&E_0(t,x,\tau,\xi;\gamma;\varepsilon) := \chi_0(t) \Theta_\gamma(\xi) P(t,x,\tau - i\gamma, \xi; \varepsilon)^{-1}, \\
&E_k(t,x,\tau,\xi;\gamma;\varepsilon) \\
&:= - \sum_{\substack{\tilde\alpha \in (\mathbf{Z}_+)^{n+1},\, |\tilde\alpha| + \mu = k \\ 0 \le \mu \le k-1}} \frac{1}{\tilde\alpha !} E_\mu^{(\tilde\alpha)}(t,x,\tau,\xi;\gamma;\varepsilon) P_{(\tilde\alpha)}(t,x,\tau - i\gamma, \xi; \varepsilon) P(t,x,\tau - i\gamma, \xi; \varepsilon)^{-1} \\
&(\ k = 1, 2, \cdots), \\
&E^N(t,x,\tau,\xi;\gamma;\varepsilon) := \sum_{k=0}^{N} E_k(t,x,\tau,\xi;\gamma;\varepsilon)
\end{aligned}$$

として

$$\begin{aligned}
&E^N(t,x,\tau,\xi;\gamma;\varepsilon) \circ P(t,x,\tau - i\gamma, \xi; \varepsilon) - \chi_0(t) \Theta_\gamma(\xi) \\
&\quad \in S(\langle \xi \rangle_\gamma^{-N-1}, g_0) \quad \text{uniformly in } \gamma \text{ and } \varepsilon
\end{aligned}$$

ここで

$$g_{0\,(t,x,\tau,\xi)} = (dt)^2 + |dx|^2 + \langle (\tau,\xi) \rangle_\gamma^{-2} (d\tau)^2 + \langle \xi \rangle_\gamma^{-2} |d\xi|^2$$

$\rightarrow \quad \exists C(l) > 0 \ (\ l \in \mathbf{R}), \exists \tilde{\nu} > 0$ s.t.

$$\sum_{\mu=0}^{m} \int_0^{6\delta_1} \| D_t^{\mu} \langle D_x \rangle^{l-\mu} v \|^2_{L^2(\mathbf{R}_x^n)} \, dt \le C(l) \int_0^{6\delta_1} \| \langle D_x \rangle^{l+\tilde{\nu}} P(t,x,D_t,D_x;\varepsilon) v \|^2_{L^2(\mathbf{R}_x^n)} \, dt$$

for $l \in \mathbf{R}$ and $v(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}_x^n))$ with $v|_{t \le 0} = 0$

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