Remarks on semi-algebraic functions

Seiichiro Wakabayashi

April 5, 2008
the second version on August 30, 2010

In this note we shall give some facts and remarks concerning “semi-algebraic functions” which we need in another paper. We think that the results given here are all well-known, but we could not find any literature in which the main results here (Theorems 5, 10 and 11 below) are given and proved.

Definition 1. Let S be a subset of $\mathbb{R}^n$. We say that S is semi-algebraic (or a semi-algebraic set (in $\mathbb{R}^n$)) if there is a finite family $\{A_{j,k}\}_{1\le j\le m, 1\le k\le r_j}$ of subsets of $\mathbb{R}^n$ such that each $A_{j,k}$ is defined by a real polynomial equation or inequality and

$$S=\bigcup_{j=1}^m\Big(\bigcap_{k=1}^{r_j} A_{j,k}\Big).$$

Noting that

$$\bigcup_{j=1}^m\Big(\bigcap_{k=1}^{r_j} A_{j,k}\Big)=\bigcap_{k_1=1}^{r_1}\cdots\bigcap_{k_m=1}^{r_m}\Big(\bigcup_{j=1}^m A_{j,k_j}\Big),$$

we have the following

Lemma 2. *Let S_1 and S_2 be semi-algebraic sets in $\mathbb{R}^n$. Then $S_1^c(=\mathbb{R}^n\setminus S_1)$, $S_1\cup S_2$ and $S_1\cap S_2$ are semi-algebraic. Moreover, if T is a semi-algebraic set in $\mathbb{R}^m$, then $S_1\times T$ is semi-algebraic.*

The following theorem is called the Tarski-Seidenberg theorem (see, *e.g.*, §A.2 of [H]).

Theorem 3 (Tarski-Seidenberg)**.** *Let S be a semi-algebraic set in $\mathbb{R}^{n+m}$. Then*

$$\widetilde{S}:=\{x\in\mathbb{R}^n;\ (x,y)\in S \text{ for some } y\in\mathbb{R}^m\}$$

is semi-algebraic.

Corollary. *Let S and T be semi-algebraic sets in $\mathbb{R}^{n+m}$ and $\mathbb{R}^m$, respectively. Then the set*

$$\widehat{S} \equiv \{x \in \mathbb{R}^n;\ (x,y) \in S \textit{ for any } y \in T\}$$

is semi-algebraic.

Proof. Since

$$\begin{aligned}\widehat{S}^c (= \mathbb{R}^n \setminus \widehat{S}) &= \{x \in \mathbb{R}^n;\ \text{there is } y \in T \text{ satisfying } (x,y) \in S^c\} \\ &= \{x \in \mathbb{R}^n;\ \text{there is } y \in \mathbb{R}^m \text{ satisfying } (x,y) \in S^c \cap (\mathbb{R}^n \times T)\},\end{aligned}$$

Theorem 3 and Lemma 2 prove the corollary. □

Lemma 4. *If S is a semi-algebraic set in $\mathbb{R}^n$, then the closure $\overline{S}$ of S and the interior $\overset{\circ}{S}$ of S are semi-algebraic.*

Remark. If $S = \{x \in \mathbb{R};\ x^2(x-1) > 0\}$, then $S = (1,\infty)$ and $\overline{S} \neq \{x \in ;\ x^2(x-1) \geq 0\} = \overline{S} \cup \{0\}$.

Proof. Put

$$\begin{aligned}D &:= \{(x,\varepsilon,y) \in \mathbb{R}^{2n+1};\ \varepsilon > 0,\ y \in S, x \in \mathbb{R}^n \text{ and } |x-y|^2 < \varepsilon\}, \\ E &:= \{(x,\varepsilon) \in \mathbb{R}^n \times (0,\infty);\ \text{there is } y \in S \text{ satisfying } |x-y|^2 < \varepsilon\}.\end{aligned}$$

Then D is semi-algebraic and

$$E = \{(x,\varepsilon) \in \mathbb{R}^{n+1};\ \text{there is } y \in S \text{ satisfying } (x,\varepsilon,y) \in D\}.$$

From Theorem 3 E is semi-algebraic. Since $\overline{S} = \{x \in \mathbb{R}^n;\ (x,\varepsilon) \in E$ for any $\varepsilon > 0\}$, and $\overset{\circ}{S} = \overline{(\mathbb{R}^n \setminus S)}^c$, $\overline{S}$ and $\overset{\circ}{S}$ are semi-algebraic. □

Theorem 5. *Let $P(X)$ be a polynomial of $X = (X_1,\cdots,X_n)$, and put $A \equiv \{X \in \mathbb{R}^n;\ P(X) \neq 0\}$. Then the number of the connected components of A is finite and each component is semi-algebraic.*

Proof. We may assume that the coefficients of $P(X)$ are real, replacing $P(X)$ by $P_{\mathrm{Re}}(X)^2 + P_{\mathrm{Im}}(X)^2$ if necessary, where $P(X) = P_{\mathrm{Re}}(X) + iP_{\mathrm{Im}}(X)$ and $P_{\mathrm{Re}}(X)$ and $P_{\mathrm{Im}}(X)$ are real polynomials. Let us prove the theorem by induction on n. If $n = 1$, then the theorem is trivial. Let $L \in \mathbb{N}(= \{1,2,\cdots\})$, and suppose that the theorem is valid when $n \leq L$. Let $n = L+1$. We can write

$$P(X) = P_1(X)^{m_1} \cdots P_s(X)^{m_s},$$

where the $P_j(X)$ are irreducible polynomials and mutually prime. Put

$$Q(X) = P_1(X) \cdots P_s(X),$$

and denote by $Q^0(X)$ the principal part (the terms of highest degree) of $P(X)$. We may assume that $Q^0(0,\cdots,0,1) \neq 0$, using linear transformation if necessary. Let $D(X')$ be the discriminant of the equation $Q(X',X_n)=0$ in X_n, where $X' = (X_1,\cdots,X_{n-1})$. Then $D(X') \not\equiv 0$ and, by the assumption of induction, there are $N \in \mathbb{N}$ and semi-algebraic sets A_j in $\mathbb{R}^{n-1}$ ($1 \leq j \leq N$) such that the A_j are mutually disjoint and coincide with the connected components of the set $\{X' \in \mathbb{R}^{n-1};\, D(X') \neq 0\}$. For each $j \in \mathbb{N}$ with $1 \leq j \leq N$ we can write

$$\begin{aligned}
&Q(X) = Q^0(0,\cdots,0,1)\prod_{k=1}^{l}(X_n - \lambda_k(X')),\\
&\lambda_1(X') < \lambda_2(X') < \cdots < \lambda_{r(j)}(X'), \quad \operatorname{Im}\lambda_k(X') \neq 0 \ (\, r(j)+1 \leq k \leq l)
\end{aligned}$$

for $X' \in A_j$, where $l = \deg_{X_n} Q(X)$ and $r(j) \in \mathbb{N}$, since the equation $Q(X',X_n)=0$ in X_n has only simple roots for $X' \in A_j$. Put

$$\begin{aligned}
A_{j,k} := \{X \in A_j \times \mathbb{R};\ &\text{there are } \lambda_1,\cdots,\lambda_{r(j)} \in \mathbb{R} \text{ and } \lambda_{r(j)+1},\cdots\lambda_l \in \mathbb{C}\\
&\text{such that } \lambda_1 < \lambda_2 < \cdots < \lambda_{r(j)},\ \operatorname{Im}\lambda_\mu \neq 0 \ (\, \mu = r(j)+1,\cdots,l),\\
&Q(X',t) = Q^0(0,\cdots,0,1)\prod_{\mu=1}^{l}(t-\lambda_\mu) \text{ as a polynomial of } t\\
&\text{and } \lambda_{k-1} < X_n < \lambda_k \text{ if } 2 \leq k \leq l,\ X_n < \lambda_1 \text{ if } k=1,\\
&\text{and } X_n > \lambda_{r(j)} \text{ if } k = r(j)+1\} \qquad (\, k = 1,2,\cdots,r(j)+1).
\end{aligned}$$

Then the $A_{j,k}$ are semi-algebraic and

$$A \cap (A_j \times \mathbb{R}) = \bigcup_{k=1}^{r(j)+1} A_{j,k}.$$

By Lemmas 2 and 4 $B_{j,k} \equiv \overline{A_{j,k}} \cap A$ is semi-algebraic. Assume that there are disjoint open subsets C_1 and C_2 of $B_{j,k}$ satisfying $B_{j,k} = C_1 \cup C_2$ and $C_2 \cap A_{j,k} \neq \emptyset$. Since $A_{j,k}$ is connected, $C_1 \subset \partial A_{j,k} \cap A$, where ∂B denotes the boundary of B in $\mathbb{R}^n$ for a subset B of $\mathbb{R}^n$. So we have $C_1 = \emptyset$. This implies that $B_{j,k}$ are connected. Since $\overline{(A_j \times \mathbb{R})} \cap A = \bigcup_{k=1}^{r(j)+1} B_{j,k}$, we have

$$A = \bigcup_{j=1}^{N} \bigcup_{k=1}^{r(j)+1} B_{j,k}.$$

Put

$$\Lambda := \{(j,k) \in \mathbb{N} \times \mathbb{N};\ 1 \leq j \leq N,\ 1 \leq k \leq r(j)+1\}.$$

For $(j,k),(j',k') \in \Lambda$ we say that $(j,k) \sim (j',k')$ if there are $\nu \in \mathbb{N}$ and $(j_\mu,k_\mu) \in \Lambda$ ($1 \le \mu \le \nu$) satisfying $B_{j_\mu,k_\mu} \cap B_{j_{\mu+1},k_{\mu+1}} \neq \emptyset$ ($0 \le \mu \le \nu$), where $(j_0,k_0) = (j,k)$ and $(j_{\nu+1},k_{\nu+1}) = (j',k')$. For $(j,k) \in \Lambda$ we put

$$A_{(j,k)} := \bigcup_{(j',k') \sim (j,k)} B_{j',k'}.$$

Then $A_{(j,k)}$ is a connected component of A and semi-algebraic. Moreover, we have $A = \bigcup_{(j,k)\in\Lambda} A_{(j,k)}$, which proves the theorem. □

Definition 6. Let $f(X)$ be a real-valued function defined on $\mathbb{R}^n$. We say that $f(X)$ is semi-algebraic (or a semi-algebraic function) if the graph of f ($= \{(X,y) \in \mathbb{R}^{n+1};\, y = f(X)\}$) is a semi-algebraic set.

Lemma 7. *$f(X)$ is semi-algebraic if and only if $A \equiv \{(X,y) \in \mathbb{R}^{n+1};\, y \le f(X)\}$ is a semi-algebraic set.*

Proof. Assume that $f(X)$ is semi-algebraic. Then $B \equiv \{(X,y,\lambda) \in \mathbb{R}^{n+2};\, \lambda = f(X)$ and $y \le \lambda\}$ is a semi-algebraic set. Therefore, Theorem 3 implies that A is semi-algebraic. Next assume that A is semi-algebraic. Then $C \equiv \{(X,y,\lambda) \in \mathbb{R}^{n+1};\, \lambda \le f(X)$ and $y < \lambda\}$ is semi-algebraic. Therefore, Theorem 3 implies that $D \equiv \{(X,y) \in \mathbb{R}^{n+1};\, y < f(X)\}$ is semi-algebraic. Thus $A \setminus D = \{(X,y) \in \mathbb{R}^{n+1};\, y = f(X)\}$ is semi-algebraic. □

Definition 8. (i) Let $f(X)$ be a complex-valued function defined on $\mathbb{R}^n$. We say that $f(X)$ is semi-algebraic (or a semi-algebraic function) if $\mathrm{Re}\, f(X)$ and $\mathrm{Im}\, f(X)$ are semi-algebraic.

(ii) Let $X^0 \in \mathbb{R}^n$, and let $f(X)$ be a complex-valued function defined in a neighborhood of X^0. We say that $f(X)$ is semi-algebraic at X^0 if there is $r > 0$ such that the sets $\{(X,y) \in \mathbb{R}^{n+1};\, |X - X^0| < r$ and $y = \mathrm{Re}\, f(X)\}$ and $\{(X,y) \in \mathbb{R}^{n+1};\, |X - X^0| < r$ and $y = \mathrm{Im}\, f(X)\}$ are semi-algebraic.

(iii) Let U be an open subset of $\mathbb{R}^n$, and let $f(X)$ be a complex-valued function defined in U. We say that $f(X)$ is semi-algebraic in U if $f(X)$ is semi-algebraic at every $X^0 \in U$.

Lemma 9. *Let $X^0 \in \mathbb{R}^n$, and let $f(X)$ and $g(X)$ be semi-algebraic (resp. semi-algebraic at X^0).*

(i) *$\alpha f(X) + \beta g(X)$ and $f(X)g(X)$ are semi-algebraic (resp. semi-algebraic at X^0), where $\alpha, \beta \in \mathbb{C}$.*

(ii) *If $g(X) \neq 0$ for $X \in \mathbb{R}^n$ (resp. $g(X) \neq 0$ in a neighborhood of X^0), then $f(X)/g(X)$ is semi-algebraic (resp. semi-algebraic at X^0).*

(iii) *If $g(X) \ge 0$ for $X \in \mathbb{R}^n$ (resp. $g(X) \ge 0$ in a neighborhood of X^0), then $g(X)^{1/l}$ (≥ 0) is semi-algebraic (resp. semi-algebraic at X^0), where $l \in \mathbb{N}$.*

Proof. Let us prove the first part of the assertion (i) in the case where $f(X)$ and $g(X)$ are semi-algebraic at X^0. The other assertions can be proved by the same argument. We may assume that $f(X)$ and $g(X)$ are real-valued. By assumption there is $r>0$ such that $A \equiv \{(X,\lambda) \in \mathbb{R}^{n+1};\ |X-X^0|<r \text{ and } \lambda = f(X)\}$ and $B \equiv \{(X,\mu) \in \mathbb{R}^{n+1};\ |X-X^0|<r \text{ and } \mu = g(X)\}$ are semi-algebraic sets. Since

$$C := \{(X,\lambda,\mu,y) \in \mathbb{R}^{n+3};\ |X-X^0|<r,\ \lambda = f(X),\ \mu = g(X) \\ \text{and } y = \alpha\lambda + \beta\mu\}$$

is semi-algebraic, Theorem 3 implies that $\alpha f(X) + \beta g(X)$ is semi-algebraic at X^0. □

Theorem 10. *Let $X^0 \in \mathbb{R}^n$, and assume that $f(X)$ is in C^∞ and semi-algebraic (resp. semi-algebraic at X^0). Then there is a irreducible polynomial $P(z,X)(\not\equiv 0)$ of $(z,X)=(z,X_1,\cdots,X_n)$ satisfying $P(f(X),X)\equiv 0$ (resp. $P(f(X),X)=0$ in a neighborhood of X^0).*

Proof. Let us prove the theorem in the case where $f(X)$ is semi-algebraic at X^0. We may assume that $f(X)$ is real-valued. By assumption there is $r>0$ such that $f(X) \in C^\infty(B_r(X^0))$ and the set $S \equiv \{(X,y) \in B_r(X^0)\times\mathbb{R};\ y = f(X)\}$ is semi-algebraic, where $B_r(X^0) = \{X \in \mathbb{R}^n;\ |X-X^0|<r\}$. First consider the case where $n=1$. Let $F(z,X)$ be the product of all polynomials $F_{j,k}(z,X)$, except polynomials depending only on X, that appear in the definition of the semi-algebraic set S in Definition 1 as $A_{j,k} = \{(z,X) \in \mathbb{R}^{n+1};\ F_{j,k}(z,X) = 0 \text{ (resp. } > 0)\}$. Then we have $F(f(X),X)=0$ in $B_r(X^0)$ since S is a graph of $f(X)$. Write

$$F(z,X) = F_1(z,X)^{m_1}\cdots F_s(z,X)^{m_s},$$

where the $F_j(z,X)$ are irreducible polynomials and mutually prime. We put

$$G(z,X) = F_1(z,X)\cdots F_s(z,X)$$

and denote by $D(X)$ the discriminant of the equation $G(z,X)=0$ in z. Then $D(X)\not\equiv 0$. Let $X^1 \in B_r(X^0)$, and assume that $D(X^1)\neq 0$. Since the roots of $G(z,X^1)=0$ in z are all simple, $f(X)$ is analytic at X^1, and there is $j(X^1)\in\mathbb{N}$ with $1\le j(X^1)\le s$ such that $F_{j(X^1)}(f(X),X)=0$ in a neighborhood of X^1. Next assume that $D(X^1)=0$. Then there is $\delta>0$ such that $D(X)\neq 0$ if $0<|X-X^1|<\delta$. Moreover, $f(X)$ is equal to a convergent Puiseux series if $0<\pm(X-X^1)<\delta$, respectively, modifying δ if necessary. Since $f(X)$ is in C^∞, the Puiseux series are Taylor series and, therefore, $f(X)$ is analytic at X^1. So $f(X)$ is analytic in $B_r(X^0)$ and there is $j\in\mathbb{N}$ with $1\le j\le s$ such that $F_j(f(X),X)=0$ in $B_r(X^0)$. Next let us consider the case where $n\ge 2$. Similarly, there is a polynomial $F(z,X)$ ($\not\equiv 0$) such that $F(f(X),X)=0$ in $B_r(X^0)$. Write

$$F(z,X) = F_1(z,X)^{m_1}\cdots F_s(z,X)^{m_s},$$

where the $F_j(z,X)$ are irreducible polynomials and mutually prime. We put

$$G(z,X) = F_1(z,X)\cdots F_s(z,X)$$

and denote by $D(X)$ the discriminant of the equation $G(z,X) = 0$ in z. We have $D(X) \not\equiv 0$. We may assume that $D^0(0,\cdots,0,1) \neq 0$, where $D^0(X)$ denotes the principal part of $D(X)$, using linear transformation if necessary. If $D(X^0) \neq 0$, then $f(X)$ is analytic at X^0 and we can choose $j \in \mathbb{N}$ with $1 \leq j \leq s$ so that $F_j(f(X),X) = 0$ in a neighborhood of X^0. Now assume that $D(X^0) = 0$. Choose $X^{1\prime} \in \mathbb{R}^{n-1}$ so that $|X^{1\prime} - X^{0\prime}| < r$, where $X^0 = (X_1^0,\cdots,X_n^0)$ and $X^{0\prime} = (X_1^0,\cdots,X_{n-1}^0)$. Since $D(X^{1\prime},X_n) \not\equiv 0$ in X_n, applying the same argument for the case $n = 1$, we can see that $f(X^{1\prime},X_n)$ is analytic in X_n if $(X^{1\prime},X_n) \in B_r(X^0)$ and that there is $j \in \mathbb{N}$ with $1 \leq j \leq s$ satisfying $F_j(f(X^{1\prime},X_n),X^{1\prime},X_n) = 0$ if $(X^{1\prime},X_n) \in B_r(X^0)$. On the other hand, for each connected component A_k of the set $\{X \in \mathbb{R}^n;\ D(X) \neq 0\}$ there is $j \equiv j(A_k) \in \mathbb{N}$ with $1 \leq j \leq s$ satisfying $F_j(f(X),X) = 0$ in $A_k \cap B_r(X^0)$. Therefore, there are $\delta > 0$ and $j \in \mathbb{N}$ such that $1 \leq j \leq s$ and $F_j(f(X),X) = 0$ if $X \in B_r(X^0)$ and $|X' - X^{0\prime}| < \delta$. □

Theorem 11. *Let $X^0 \in \mathbb{R}^n$, and assume that $f(X)$ is a continuous function defined on $\mathbb{R}^n$ (resp. near X^0). Moreover, we assume that there is a polynomial $P(z,X)$ ($\not\equiv 0$) satisfying $P(f(X),X) \equiv 0$ (resp. $P(f(X),X) = 0$ in a neighborhood of X^0). Then $f(X)$ is semi-algebraic (resp. semi-algebraic at X^0).*

Proof. Let us prove the theorem in the case where $f(X)$ is defined in $B_r(X^0)$. We may assume that $f(X)$ is real-valued and that $P(z,\lambda)$ is a real polynomial. Write

$$P(z,X) = P_1(z,X)^{m_1}\cdots P_s(z,X)^{m_s},$$

where the $P_j(z,X)$ are irreducible and mutually prime. We put

$$Q(z,X) = P_1(z,X)\cdots P_s(z,X)$$

and denote by $D(X)$ the discriminant of the equation $Q(z,X) = 0$ in z. Then we have $D(X) \not\equiv 0$. Put $A := \{X \in \mathbb{R}^n;\ D(X) \neq 0\}$. It follows from Theorem 5 that there are a finite number of semi-algebraic sets $A_1, \cdots, A_N$ in $\mathbb{R}^n$ such that the A_j are the disjoint connected components of A and $A = \bigcup_{j=1}^N A_j$. For each $j \in \mathbb{N}$ with $1 \leq j \leq N$ there are $r(j) \in \mathbb{N}$ with $1 \leq r(j) \leq m$, a polynomial $c(X)$ and $\lambda_k(X)$ defined in A_j ($1 \leq k \leq m$) such that $c(X) \neq 0$ and

$$Q(z,X) = c(X)\prod_{k=1}^{m}(z - \lambda_k(X))$$

$$\lambda_1(X) < \lambda_2(X) < \cdots < \lambda_{r(j)}(X), \quad \operatorname{Im}\lambda_k(X) \neq 0 \ (r(j)+1 \leq k \leq m)$$

for $X \in A_j$, where $m = \deg_z Q(z,X)$. Let $j \in \mathbb{N}$ satisfy $1 \le j \le N$ and $A_j \cap B_r(X^0) \neq \emptyset$. Then there exists uniquely $k(j) \in \mathbb{N}$ satisfying $1 \le k(j) \le r(j)$ and $\lambda_{k(j)}(X) = f(X)$ in $A_j \cap B_r(X^0)$. Put

$$
\begin{aligned}
E_j := \{(X,y) \in A_j \times \mathbb{R};\ & X \in B_r(X^0) \text{ and there are } a \in \mathbb{R} \text{ and } \lambda_1, \cdots, \lambda_m \in \mathbb{C} \\
& \text{such that } Q(z,X) = a \prod_{k=1}^{m} (z - \lambda_k),\ \lambda_1 < \cdots < \lambda_{r(j)}, \\
& \operatorname{Im} \lambda_k \neq 0\ (\ r(j)+1 \le k \le m) \text{ and } y = \lambda_{k(j)}\}.
\end{aligned}
$$

Then E_j is semi-algebraic and

$$E_j = \{(X,y) \in A_j \times \mathbb{R};\ X \in B_r(X^0) \text{ and } y = f(X)\}.$$

Put

$$\widetilde{E}_j := \{(X,y) \in \overline{A_j} \times \mathbb{R};\ X \in B_r(X^0) \text{ and } y = f(X)\}.$$

Since $\widetilde{E}_j = \overline{E_j} \cap B_r(X^0) \times \mathbb{R}$, $\widetilde{E}_j$ is semi-algebraic. So $E \equiv \bigcup_{j: A_j \cap B_r(X^0) \neq \emptyset} \widetilde{E}_j$ is semi-algebraic. Note that $\bigcup_{j=1}^{N} \overline{A_j} = \mathbb{R}^n$ and that $\overline{A_j} \cap B_r(X^0) = \emptyset$ if $A_j \cap B_r(X^0) = \emptyset$. Then we have

$$E = \{(X,y) \in B_r(X^0) \times \mathbb{R};\ y = f(X)\}.$$

□

References

[H] L. Hörmander, The Analysis of Linear Partial Differential Operators II, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.