Remarks on semi-algebraic functions II

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September 6, 2010

This note is a supplement to [W]. In this note we slightly modify the definition of semi-algebraic functions as follows.

Definition 1. (i) Let U be a semi-algebraic set in $\mathbb{R}^n$, and let $f(X)$ be a real-valued function defined in U. We say that $f(X)$ is semi-algebraic in U if the graph of f ($=\{(X,y)\in U\times\mathbb{R};\ y=f(X)\}$) is a semi-algebraic set.
(ii) Let $X^0\in\mathbb{R}^n$, and let $f(X)$ be a real-valued function defined in a neighborhood of X^0. We say that $f(X)$ is semi-algebraic at X^0 if there is $r>0$ such that $f(X)$ is semi-algebraic in $B_r(X^0)\equiv\{X\in\mathbb{R}^n;\ |X-X^0|<r\}$.
(iii) When $f(x)$ is a complex-valued function, we say that $f(X)$ is semi-algebraic in U (resp. at X^0) if $\operatorname{Re} f(X)$ and $\operatorname{Im} f(X)$ are semi-algebraic in U (resp. at X^0).

Lemma 2. *Let $m,n\in\mathbb{Z}_+$, and let S and T be semi-algebraic sets in $\mathbb{R}^{n+m}$. For $X\in\mathbb{R}^n$ we define*

$$T(X)=\{Y\in\mathbb{R}^n;\ (X,Y)\in T\}.$$

Then the set

$$A\equiv\{X\in\mathbb{R}^n;\ (X,Y)\in S \text{ for } \forall Y\in T(X)\}$$

is a semi-algebraic set in $\mathbb{R}^n$.

Remark. Let U be a semi-algebraic set in $\mathbb{R}^n$. Then $\{X\in U;\ (X,Y)\in S$ for $\forall Y\in T(X)\}$ is semi-algebraic.

Proof. We have

$$\begin{aligned}A^c(=\mathbb{R}^n\setminus A)&=\{X\in\mathbb{R}^n;\ \exists Y\in T(X)\ \text{s.t.}\ (X,Y)\in S^c\}\\&=\{X\in\mathbb{R}^n;\ \exists Y\in\mathbb{R}^m\ \text{s.t.}\ (X,Y)\in T\cap S^c\}.\end{aligned}$$

From Lemma 2 in [W] $T\cap S^c$ is semi-algebraic. So the Tarski-Seidenberg Theorem implies that A^c is semi-algebraic (see, *e.g.*, Theorem 3 in [W]). Thus A is semi-algebraic. □

Theorem 3. *Let U be a semi-algebraic set in $\mathbb{R}^n$, and let $t(X)$ be a semi-algebraic function in U satisfying $t(X) > 0$. Put*

$$\Omega = \{(X,t) \in U \times \mathbb{R};\ 0 < t < t(X)\},$$

and let $f(X,t)$ be a real-valued semi-algebraic function in Ω. If $g(X) \equiv \lim_{t \downarrow 0} f(X,t)$ exists for $X \in U$, then $g(X)$ is semi-algebraic in U.

Proof. By definition $G \equiv \{(X,t,y) \in \Omega \times \mathbb{R};\ y = f(X,t)\}$ is semi-algebraic.

$$\begin{aligned} A =&\{(X,t,y,\varepsilon,\delta,f) \in \mathbb{R}^{n+5};\ X \in U,\ \varepsilon > 0,\ 0 < \delta \le t(X), \\ &0 < t < \delta \text{ and } (X,t,f) \in G\}. \end{aligned}$$

Then A is semi-algebraic. For $X \in U$, $y \in \mathbb{R}$, $\varepsilon > 0$ and $\delta \in (0, t(X)]$ we define

$$A(X,y,\varepsilon,\delta) = \{(t,f) \in \mathbb{R}^2;\ (X,t,y,\varepsilon,\delta,f) \in A\}.$$

Moreover, we put

$$\begin{aligned} B &= \{(X,y,\varepsilon,\delta) \in \mathbb{R}^{n+3};\ X \in U,\ \varepsilon > 0,\ \delta \in (0,t(X)] \text{ and} \\ &\quad (f-y)^2 \le \varepsilon^2 \text{ for } \forall (t,f) \in A(X,y,\varepsilon,\delta)\} \\ C &= \{(X,y,\varepsilon) \in \mathbb{R}^{n+2};\ \exists \delta \in \mathbb{R} \text{ s.t. } (X,y,\varepsilon,\delta) \in B\}, \\ D &= \{(X,y) \in \mathbb{R}^{n+1};\ (X,y,\varepsilon) \in C \text{ for } \forall \varepsilon > 0\}. \end{aligned}$$

From Lemma 2 (or its remark) it follows that B is semi-algebraic and, therefore, C is semi-algebraic by the Tarski-Seidenberg theorem. Moreover, it follows from Corollary of Theorem 3 in [W] that D is semi-algebraic. On the other hand, we have

$$D = \{(X,y) \in \mathbb{R}^{n+1};\ X \in U \text{ and } y = g(X)\}.$$

Indeed, for each $X \in U$ and any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(X,t) - y| < \varepsilon \quad \text{for any } t \in (0,\delta)$$

and, therefore, $y = g(X)$, if $(X,y) \in D$. It is obvious that $(X, g(X)) \in D$ if $X \in U$. So $g(X)$ is semi-algebraic in U. □

Corollary. *Let U be an open semi-algebraic set in $\mathbb{R}^n$, and let $f(X)$ be real-valued and semi-algebraic in U. Assume that $(\partial/\partial X_1)f(X)$ exists for $x \in U$. Then $(\partial/\partial X_1)f(X)$ is semi-algebraic in U.*

Proof. Put

$$E = \{(X,\delta) \in U \times (0,1];\ B_\delta(X) \subset U\}.$$

It is obvious that $E \cap \{X\} \times \mathbb{R} \neq \emptyset$ for each $X \in E$. We define

$$t(X) = \sup\{\delta;\ (X,\delta) \in E\}.$$

$t(X)$ is semi-algebraic in U (see, *e.g.*, Corollary A.2.4 of [H]). Put

$$f(X,t) = \frac{1}{t}(f(X+te_1) - f(X)),$$

where $e_1 = (1,0,\cdots,0) \in \mathbb{R}^n$. Appying Theorem 3 we complete the proof, since $\lim_{t\downarrow 0} f(X,t) = (\partial/\partial X_1)f(X)$. □

Lemma 4. *Let I be an interval of $\mathbb{R}$, and let $F(t)$ be real analytic and semi-algebraic in I. Then the set $A \equiv \{t \in I;\ F(t) = 0\}$ is finite.*

Proof. Since A is semi-algebraic, A is defined by a finite family $\{A_j;\ 1 \le j \le M\}$ of semi-algebraic subsets of $\mathbb{R}$, where $A_j = \{t \in \mathbb{R};\ p_j(t) = 0\}$ or $A_j = \{t \in \mathbb{R};\ p_j(t) > 0\}$ with polynomials $p_j(t)(\not\equiv 0)$ ($1 \le j \le M$). Suppose that there is $t_0 \in A$ satisfying $p_j(t_0) \neq 0$ for any j. Then there is $\delta > 0$ satisfying $(t_0 - \delta, t_0 + \delta) \subset A$, which contradicts discreteness of the set A. Therefore, we have

$$A \subset \bigcup_{j=1}^{M} \{t \in \mathbb{R};\ p_j(t) = 0\}$$

which implies that A is finite. □

Theorem 5. *Let I be an interval of $\mathbb{R}$, and assume that $a_j(t) \in C^\infty(I)$ ($1 \le j \le m$) are semi-algebraic in I, where $m \in \mathbb{N}$. If $\lambda(t) \in C(I)$ satisfies*

$$\lambda(t)^m + a_1(t)\lambda(t)^{m-1} + \cdots + a_m(t) = 0 \quad \textit{in } I,$$

then $\lambda(t)$ is semi-algebraic in I.

Proof. There are $m' \in \mathbb{N}$ and semi-algebraic functions $\tilde{a}_j(t)$ in I ($1 \le j \le m'$) such that the $\tilde{a}_j(t) \in C^\infty(I)$ and

$$(\mathrm{Re}\,\lambda(t))^{m'} + \tilde{a}_1(t)(\mathrm{Re}\,\lambda(t))^{m'-1} + \cdots + \tilde{a}_{m'}(t) = 0 \quad \text{in } I.$$

Here the $\tilde{a}_j(t)$ are given as polynomials of $a_1(t)$, $\overline{a_1(t)}$, $\cdots$, $a_m(t)$, $\overline{a_m(t)}$. For $\mathrm{Im}\,\lambda(t)$ we have the same. So we may assume that $\lambda(t)$ is real-valued. Moreover, we may assume that the $a_j(t)$ are real-valued. From the proof of Theorem 10 in [W] we see that the $a_j(t)$ are real analytic in I. We define

$$\mathscr{B} = \{a(t);\ a(t) \text{ is a complex-valued semi-algebraic function}$$

defined in I and real analytic in $I\}$.

It follows from Lemma 9 in [W] (or its proof) that $\mathscr{B}$ is a subring of $\mathscr{A}(I)$, where $\mathscr{A}(I)$ denotes the space of real analytic functions defined in I, We denote by $\widetilde{\mathscr{B}}$ the quotient field of $\mathscr{B}$. Write

$$P(\lambda,t)=\lambda^m+a_1(t)\lambda^{m-1}+\cdots+a_m(t)\in\mathscr{B}[\lambda]\subset\widetilde{\mathscr{B}}[\lambda].$$

Then there are $s\in\mathbb{N}$, $m_j\in\mathbb{N}$ and irreducible polynomials $P_j(\lambda,t)\in\widetilde{\mathscr{B}}[\lambda]$ ($1\le j\le s$) such that $P_1(\lambda,t)$, $\cdots$, $P_s(\lambda,t)$ are mutually prime and

$$P(\lambda,t)=P_1(\lambda,t)^{m_1}\cdots P_s(\lambda,t)^{m_s}.$$

We note that the $P_j(\lambda,t)$ can be chosen in $\mathscr{B}[\lambda]$ (see, *e.g.*, IV§6 of [L]). Put

$$Q(\lambda,t)=P_1(\lambda,t)\cdots P_s(\lambda,t),$$

and denote by $D(t)$ the discriminant of $Q(\lambda,t)=0$ in λ. Then we have $D(t)\not\equiv 0$ in I, since $Q(\lambda,t)$ and $(\partial/\partial\lambda)Q(\lambda,t)$ are mutually prime. By Lemma 4 we can write

$$\{t\in I;\ D(t)=0\}=\{\tau_1,\tau_2,\cdots,\tau_N\},\quad \tau_1<\tau_2<\cdots<\tau_N.$$

Put

$$I_0=(-\infty,\tau_1)\cap I,\ \ I_1=(\tau_1,\tau_2),\ \ \cdots,\ \ I_{N-1}=(\tau_{N-1},\tau_N),\ \ I_N=(\tau_N,\infty)\cap I.$$

Then $Q(\lambda,t)=0$ in λ has only simple roots for $0\le j\le N$ and $t\in I_j$. We fix $j\in\{0,1,\cdots,N\}$. For $t\in I_j$ we can write

$$\begin{aligned}&Q(\lambda,t)=\prod_{k=1}^{\hat{m}}(\lambda-\lambda_{j,k}(t)),\\&\lambda_{j,1}(t)<\lambda_{j,2}(t)<\cdots<\lambda_{j,r(j)}(t),\quad \operatorname{Im}\lambda_{j,k}(t)\neq 0\ (\ r(j)+1\le k\le\hat{m}),\end{aligned}$$

where $\hat{m}=\deg_\lambda Q(\lambda,t)$ and $1\le r(j)\le\hat{m}$. By assumption there is $k(j)\in\mathbb{N}$ such that $1\le k(j)\le r(j)$ and $\lambda(t)=\lambda_{j,k(j)}(t)$ for $t\in I_j$. Put

$$\begin{aligned}&E=\{(z,t,Q(z,t))\in\mathbb{R}^3;\ t\in I\}\\&F_j=\{(t,y)\in I_j\times\mathbb{R};\ \exists\lambda_1,\cdots,\lambda_{\hat{m}}\in\mathbb{C}\ \text{ s.t.}\\&\qquad\Big(z,t,\prod_{k=1}^{\hat{m}}(z-\lambda_k)\Big)\in E\ \text{for}\ \forall z\in\mathbb{R},\ \ \lambda_1<\lambda_2<\cdots<\lambda_{r(j)},\\&\qquad\operatorname{Im}\lambda_k\neq 0\ (\ r(j)+1\le k\le\hat{m})\ \text{ and }\ y=\lambda_{j,k(j)}\}.\end{aligned}$$

It is obvious that E and F_j are semi-algebraic and

$$F_j = \{(t,y) \in I_j \times \mathbb{R};\ y = \lambda(t)\},$$

which implies that $\lambda(t)$ is semi-algebraic in I_j. Since $\bigcup_{j=1}^N \{(\tau_j, \lambda(\tau_j))\} \cup \bigcup_{j=0}^N F_j$ is semi-algebraic, $\lambda(t)$ is semi-algebraic in I. $\square$

I could not prove Theorem 5 when I is an open connected semi-algebraic subset of $\mathbb{R}^n$. Under stronger assumptions we have the following

Theorem 6. *Let U be an open semi-algebraic set in $\mathbb{R}^n$, and assume that U is connected, and that $a_j(X)$ ($1 \le j \le m$) are real analytic and semi-algebraic in U, where $m \in \mathbb{N}$. Put*

$$P(\lambda, X) = \lambda^m + a_1(X)\lambda^{m-1} + \cdots + a_m(X).$$

Then $\lambda(X)$ is semi-algebraic in U if $\lambda(X)$ is real analytic in U and $P(\lambda(X), X) \equiv 0$ in U.

Proof. We may assume that $\lambda(X)$ is real-valued and that the $a_j(X)$ are real-valued (see the proof of Theorem 5). Let $X^0 \in U$, and denote by $\mathscr{A}$ the set of germs of real analytic functions at X^0. Then there are $s \in \mathbb{N}$, $m_j \in \mathbb{N}$ and irreducible polynomials $P_j(\lambda, X) \in \mathscr{A}[\lambda]$ ($1 \le j \le s$) such that $P_1(\lambda, X)$, $\cdots$, $P_s(\lambda, X)$ are mutually prime and

$$P(\lambda, X) = P_1(\lambda, X)^{m_1} \cdots P_s(\lambda, X)^{m_s}.$$

Put

$$Q(\lambda, X) = P_1(\lambda, X) \cdots P_s(\lambda, X),$$

and denote by $D(X)$ the discriminant of $Q(\lambda, X) = 0$ in λ. We choose a neighborhood V of X^0 in U so that $D(X)$ is defined in V. Since $D(X) \not\equiv 0$ in V, there are $X^1 \in V$ and $\delta > 0$ such that $B_\delta(X^1) \subset V$ and $D(X) \ne 0$ for $X \in B_\delta(X^1)$. Then $Q(\lambda, X) = 0$ in λ has only simple roots for $X \in B_\delta(X^1)$. For $X \in B_\delta(X^1)$ we can represent

$$Q(\lambda, X) = \prod_{k=1}^{\hat{m}} (\lambda - \lambda_k(X)),$$
$$\lambda_1(X) < \lambda_2(X) < \cdots < \lambda_r(X), \quad \operatorname{Im} \lambda_k(X) \ne 0 \ (r+1 \le k \le \hat{m}),$$

where $\hat{m} = \deg_\lambda Q(\lambda, X)$ and $1 \le r \le \hat{m}$. By assumption there is $k_0 \in \mathbb{N}$ such that $1 \le k_0 \le r$ and $\lambda(X) = \lambda_{k_0}(X)$ in $B_\delta(X^1)$. There are $l_k \in \mathbb{N}$ ($1 \le k \le \hat{m}$) such that

$$P(\lambda, X) = \prod_{k=1}^{\hat{m}} (\lambda - \lambda_k(X))^{l_k} \quad \text{for } X \in B_\delta(X^1).$$

By Lemma 9 in [W] (or its proof) $E \equiv \{(z,X,P(z,X)) \in \mathbb{R}^{n+2};\, X \in B_\delta(X^1)\}$ is semi-algebraic. Define

$$\begin{aligned} F = \{(X,y) \in B_\delta(X^1) \times \mathbb{R};\ &\exists \lambda_1,\cdots,\lambda_{\hat{m}} \in \mathbb{C} \ \text{ s.t.} \\ &\Big(z,X,\prod_{k=1}^{\hat{m}}(z-\lambda_k)^{l_k}\Big) \in E \text{ for } \forall z \in \mathbb{R},\ \ \lambda_1 < \lambda_2 < \cdots < \lambda_r, \\ &\operatorname{Im}\lambda_k \neq 0 \ (\ r+1 \le k \le \hat{m}) \ \text{ and } \ y = \lambda_{k_0}\}. \end{aligned}$$

Then F is semi-algebraic and

$$F = \{(X,y) \in B_\delta(X^1) \times \mathbb{R};\ y = \lambda(X)\},$$

which implies $\lambda(X)$ is semi-algebraic at X^1. It follows from Theorem 10 in [W] (or its proof) that there is an irreducible polynomial $\widetilde{P}(z,X)(\not\equiv 0)$ of (z,X) satisfying $\widetilde{P}(\lambda(X),X) \equiv 0$ near X^1. Since $\lambda(X)$ is real analytic in U, by analytic continuation we have $\widetilde{P}(\lambda(X),X) \equiv 0$ in U. Theorem 11 in [W] (or its proof) implies that $\lambda(X)$ is semi-algebraic in U. □

References

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