Puiseux expansions of the roots of the equations of pseudo-polynomials with a small parameter

Seiichiro Wakabayashi

January 16, 2019

This note is a supplement to [W]. Let U be a non-void bounded open subset of $\mathbf{R}^n$, and let $m \in \mathbf{N}$ and $a_j(s,\eta) \in \mathcal{A}([-1,1] \times \overline{U})$ ($0 \le j \le m-1$). Here $f(\eta) \in \mathcal{A}(\Omega)$ means that $f(\eta)$ is a real analytic function of η in a neighborhood of Ω, where $\Omega \subset \mathbf{R}^n$. We may assume that U is connected. Put $a_m(s,\eta) \equiv 1$. Since $\overline{U}$ is compact, there is $\delta > 0$ such that the $a_j(s,\eta)$ can be regarded as analytic functions defined in a neighborhood of $V \equiv \{(s,\eta) \in \mathbf{C} \times \overline{U};\ \operatorname{Re} s \in (-1-\delta, 1+\delta)$ and $|\operatorname{Im} \delta| < \delta\}$. Define

$$p(t,s,\eta) = \sum_{j=0}^{m} a_j(s,\eta) t^j.$$

Noting that $\mathcal{A}([-1,1] \times \overline{U})$ is an integral domain, we denote by $\mathcal{K}$ its quotient field. Then we can write

$$p(t,s,\eta) = p_1(t,s,\eta)^{k_1} \cdots p_\nu(t,s,\eta)^{k_\nu},$$

where $p_j(t,s,\eta) \in \mathcal{K}[t]$ ($1 \le j \le \nu$) are monic irreducible polynomials of t, $\deg_t p_j(t,s,\eta) \ge 1$ and the $p_j(t,s,\eta)$ are mutually prime in $\mathcal{K}[t]$. Put

$$P(t,s,\eta) = p_1(t,s,\eta) \cdots p_\nu(t,s,\eta),$$

and denote by $R(s,\eta)$ the discriminant of $P(t,s,\eta) = 0$ in t, *i.e.*, the resultant of $P(t,s,\eta)$ and $(\partial P/\partial t)(t,s,\eta)$ as polynomials of t. Then we have $R(s,\eta) \ne 0$ in $\mathcal{K}$, *i.e.*, $R(s,\eta) \not\equiv 0$ in (s,η). Since $R(s,\eta) \in \mathcal{K}$, there is $\psi(s,\eta) \in \mathcal{A}([-1,1] \times \overline{U})$ such that $\psi(s,\eta) \not\equiv 0$ and $\psi(s,\eta)R(s,\eta) \in \mathcal{A}([-1,1]\times\overline{U})$. Putting $\tilde{\varphi}(s,\eta) := \psi(s,\eta)^2 R(s,\eta)$, we see that the roots of $P(t,s,\eta) = 0$ in t are all simple for $(s,\eta) \in \widetilde{V}_0 \equiv \{(s,\eta) \in V; \tilde{\varphi}(s,\eta) \neq 0\}$, modifying δ if necessary. This implies that the multiplicities of the roots of $p(t,s,\eta) = 0$ are constant for $(s,\eta) \in \widetilde{V}_0$ (see, also, Lemma in [W]). Choose $\tilde{\psi}(s,\eta) \in \mathcal{A}([-1,1]\times\overline{U})$ so that $\tilde{\psi}(s,\eta) \not\equiv 0$ and the coefficients of $\tilde{\psi}(s,\eta)P(t,s,\eta)$ belong to $\mathcal{A}([-1,1]\times\overline{U})$, and put $\varphi(s,\eta) = \tilde{\varphi}(s,\eta)\tilde{\psi}(s,\eta)$. We also modify $\delta > 0$ so that the coefficients of $\tilde{\psi}(s,\eta)P(t,s,\eta)$ and $\varphi(s,\eta)$ are analytic in V, if necessary. By the implicit function theorem there are analytic functions $t_k(s,\eta)$ ($1 \le k \le \bar{m}$) defined in $\widehat{V} \equiv \{(s,\eta) \in V;\ \varphi(s,\eta) \neq 0$ and $s \notin (-\infty,0)\}$ satisfying

$$P(t,s,\eta) = \prod_{k=1}^{\bar{m}} (t - t_k(s,\eta)),$$

where $\bar{m} = \deg_t P(t,s,\eta)$. So there are $r_k \in \{k_1,\cdots,k_\nu\}$ ($1 \le k \le \bar{m}$) such that

$$p(t,s,\eta) = \prod_{k=1}^{\bar{m}} (t - t_k(s,\eta))^{r_k} \quad \text{for } (s,\eta) \in \widehat{V}.$$

Write

$$\varphi(s,\eta) = \sum_{j=l_0}^{\infty} \varphi_j(\eta) s^j$$

which converges in $V \cap \{|s| < \delta\}$, where $\varphi_{l_0}(\eta) \not\equiv 0$. Let U_0 be a connected open subset of $\overline{U}$ satisfying $\varphi_{l_0}(\eta) \neq 0$ for $\eta \in \overline{U}_0$. Modifying $\delta > 0$ we may assume that

$$\varphi(s,\eta) \neq 0 \quad \text{in a neighborhood of } V_0 \equiv \{(s,\eta) \in \mathbf{C} \times \overline{U}_0;\ 0 < |s| < \delta\}.$$

Then the $t_k(s,\eta)$ are analytic in a neighborhood of $V_0 \cap \{s \notin (-\infty,0)\}$. For a fixed $\eta \in \overline{U}_0$ analytic continuations in s of $t_k(s,\eta)$ around $s = 0$ (crossing the negative real axis anti-clockwise) show that there are $\nu_k \equiv \nu_k(\eta) \in \mathbf{N}$ ($1 \le k \le \bar{m}$) such that

$$\begin{aligned} &t_k(se^{2\pi\nu i},\eta) \not\equiv t_k(s,\eta) \quad \text{in } s \in (0,\delta) \qquad \text{if } \nu \in \mathbf{N} \text{ and } \nu < \nu_k, \\ &t_k(se^{2\pi\nu_k i},\eta) = t_k(s,\eta) \quad (0 < s < \delta). \end{aligned}$$

Applying the same argument as in the proof of the monodromy theorem, we can prove that $\nu_k(\eta)$ does not depend on $\eta \in U_0$ (see, *e.g.*, [A]). It follows from Riemann's theorem on removable singularities that $t_k(z^{\nu_k},\eta)$ ($1\le k\le \bar m$) are (single-valued) analytic functions of (z,η) in $\{(z,\eta)\in \mathbf{C}\times \overline{U}_0;\ |z|<\delta^{1/\nu_k}\}$. Therefore, we have

$$t_k(z^{\nu_k},\eta)=\sum_{j=0}^{\infty} t_{k,j}(\eta)z^j \quad (1\le k\le \bar m),$$

in $\{(z,\eta)\in \mathbf{C}\times \overline{U}_0;\ |z|<\delta^{1/\nu_k}\}$, which are convergent, where the $t_{k,j}(\eta)$ are real analytic in $\overline{U}_0$. This implies that the $t_k(s,\eta)$ can be expanded into Puiseux series in s which converge for $(s,\eta)\in(0,\delta)\times\overline{U}_0$, *i.e.*,

$$t_k(s,\eta)=\sum_{j=0}^{\infty} t_{k,j}(\eta)s^{j/\nu_k} \quad \text{for } (s,\eta)\in(0,\delta)\times\overline{U}_0.$$

Thus we have the following

Theorem. *There are* $\varphi_0(\eta)(\not\equiv 0)\in\mathcal{A}(\overline{U})$, $\bar m\in\mathbf{N}$ *and* $r_k\in\mathbf{N}$ ($1\le k\le\bar m$) *satisfying the following*:
For each connected open subset U_0 *of* $\overline{U}$ *satisfying* $\varphi_0(\eta)\ne 0$ *for* $\eta\in\overline{U}_0$ *there are* $\delta>0$ *and* $t_k(s,\eta)\in\mathcal{A}((0,\delta)\times\overline{U}_0)$ ($1\le k\le\bar m$) such that

$$t_k(s,\eta)\ne t_l(s,\eta) \quad \text{for } (s,\eta)\in(0,\delta)\times\overline{U}_0 \quad \text{if } 1\le k<l\le\bar m,$$

$$p(t,s,\eta)=\prod_{k=1}^{\bar m}(t-t_k(s,\eta))^{r_k} \quad \text{for } (s,\eta)\in(0,\delta)\times\overline{U}_0$$

and the $t_k(s,\eta)$ *can be expanded into Puiseux series in* s *which converge for* $(s,\eta)\in(0,\delta)\times\overline{U}_0$, *i.e.*,

$$t_k(s,\eta)=\sum_{j=0}^{\infty} t_{k,j}(\eta)s^{j/\nu_k} \quad \text{for } (s,\eta)\in(0,\delta)\times\overline{U}_0,$$

where $t_{k,j}(\eta)\in\mathcal{A}(\overline{U}_0)$ *and* $\nu_k\in\mathbf{N}$ ($1\le k\le\bar m$, $j\in\mathbf{Z}_+$).

References

[A] Lars V. Ahlfors, Comlex Analysis, Third Edition,1979, McGraw-Hill Kogakusha, Ltd.

[W] Seiichiro Wakabayashi, Asymptotic expansions of the roots of the equations of pseudo-polynomials with a small parameter, located in https://wkbysh.sakura.ne.jp/index.html.