TSUKUBA J. MATH.
Vol. 49 No. 1 (2025), 1–50

ON THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS WHOSE COEFFICIENTS DEPEND ONLY ON THE TIME VARIABLE I

By

Seiichiro WAKABAYASHI

Abstract. In this paper we investigate the Cauchy problem for hyperbolic operators with triple characteristics whose coefficients depend only on the time variable. And we give sufficient conditions for C^{∞} well-posedness. We shall also consider necessary conditions in [7].

1. Introduction

In [6] we studies the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time-dependent coefficients. And we gave sufficient conditions for the Cauchy problem to be C^{∞} well-posed under the assumption that the coefficients, for instance, are real analytic. These sufficient conditions are also necessary if the space dimension is less than 3, or if the coefficients are semi-algebraic functions with respect to the time variable. In [5] we considered the Cauchy problem for hyperbolic operators of third order with time-dependent coefficients and defined the sub-sub-principal symbols. We showed that the Cauchy problem is C^{∞} well-posed under some conditions on the sub-principal symbols and the sub-sub-principal symbols. In this paper we shall deal with hyperbolic operators with time-dependent coefficients and triple characteristics and give suficient conditions for the Cauchy problem to be C^{∞} well-posed. Our results are extensions of the results given in [5] to higher-order

2020 *Mathematics Subject Classification*: Primary 35L30; Secondary 35L25.
Key words and phrases: Cauchy problem, hyperbolic, C^{∞} well-posed, triple characteristics.
This research was partially supported by Grant-in-Aid for Scientific Research (No. 16K05222), Japan Society for the Promotion of Science.
Received October 25, 2023.
Revised March 21, 2024.

hyperbolic operators. In doing so, we shall introduce new quantities as generalizations of "sub-sub-principal symbols." In [7] it will be proved that our sufficient conditions are also necessary for C^∞ well-posedness under additional conditions.

Let $m \in \mathbf{N}$ and $P(t,\tau,\xi) \equiv \tau^m + \sum_{j=1}^m \sum_{|\alpha| \le j} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1,\ldots,\xi_n)$ of degree m whose coefficients $a_{j,\alpha}(t)$ belong to $C^\infty([0,\infty])$. Here $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbf{Z}_+)^n$ is a multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$ $(= \{0,1,2,3,\ldots\})$. We consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty)\times\mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1) \end{cases}$$

in the framework of the space of C^∞ functions, where $D_t = -i\partial/\partial t$ $(= -i\partial_t)$, $D_x = (D_1,\ldots,D_n) = -i(\partial/\partial x_1,\ldots,\partial/\partial x_n)$, $f(t,x) \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j(x) \in C^\infty(\mathbf{R}^n)$ $(0 \le j \le m-1)$.

DEFINITION 1.1. (i) The Cauchy problem (CP) is said to be C^∞ well-posed if the following conditions (E) and (U) are satisfied:

(E) For any $f \in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)$ $(0 \le j \le m-1)$ there is $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$ satisfying (CP).

(U) If $s > 0$, $u \in C^\infty([0,\infty)\times\mathbf{R}^n)$, $D_t^j u(t,x)|_{t=0} = 0$ $(0 \le j \le m-1)$ and $P(t,D_t,D_x)u(t,x)$ vanishes for $t < s$, then $u(t,x)$ also vanishes for $t < s$.

(ii) We say that the Cauchy problem (CP) has finite propagation property (has finite propagation speeds) if the following condition (F) is satisfied:

(F) For any $T > 0$ there is a convex closed cone Γ_T in $\mathbf{R}^n$ (with its vertex at the origin) such that $\Gamma_T \subset \{t > 0\} \cup \{0\}$, and for any $(t_0,x^0) \in \mathbf{R}^{n+1}$ with $0 < t_0 \le T$

$$u = 0 \quad \text{in } \Gamma_T(t_0,x^0) \equiv \{(t_0,x^0)\} - \Gamma_T$$

if $u \in C^\infty(\mathbf{R}^{n+1})$, $\operatorname{supp} u \subset [0,\infty)\times\mathbf{R}^n$ and

$$P(t,D_t,D_x)u = 0 \quad \text{in } \Gamma_T(t_0,x^0).$$

We assume that the following conditions are satisfied:

(A-1) $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j, j-1)$ are real analytic on $[0,\infty)$.

(A-2) For some $\kappa_0 \in [1,3/2)$ $a_{j,\alpha}(t) \in \mathscr{E}^{\{\kappa_0\}}([0,\infty))$ $(2 \le j \le m,\ |\alpha| = j-2)$.

Here we say that $a(t) \in \mathscr{E}^{\{\kappa\}}(I)$ if for any $T > 0$ there are $h > 0$ and $C_T > 0$ satisfying

$$|\partial_t^k a(t)| \le C_T h^k (k!)^\kappa \quad \text{for } k \in \mathbf{Z}_+ \text{ and } t \in I \text{ with } |t| \le T,$$

where $1 \le \kappa < \infty$ and I is a closed interval of $\mathbf{R}$. From (A-1) there are a complex neighborhood Ω of $[0, \infty)$ (in $\mathbf{C}$) and $\delta_0 > 0$ such that $[-\delta_0, \infty) \subset \Omega$, $\Omega \cap \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \le T\}$ is compact for any $T > 0$, and $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j)$ are regarded as analytic functions defined in Ω. Applying the results in [3], we shall prove that (CP) has finite propagation property. So we assume (A-2) in order to apply the results in [3]. Put

$$p(t, \tau, \xi) = \tau^m + \sum_{j=1}^{m} \sum_{|\alpha|=j} a_{j,\alpha}(t) \tau^{m-j} \xi^\alpha \ (\equiv P_m(t, \tau, \xi)),$$

$$P_k(t, \tau, \xi) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t) \tau^{m-j} \xi^\alpha \quad (0 \le k \le m-1).$$

We also assume that the following conditions (H) and (T) are satisfied:

(H) $p(t, \tau, \xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0) \in \mathbf{R}^{n+1}$ for $t \in [-\delta_0, \infty)$, *i.e.*,

$$p(t, \tau - i, \xi) \ne 0 \quad \text{for any } (t, \tau, \xi) \in [-\delta_0, \infty) \times \mathbf{R} \times \mathbf{R}^n.$$

(T) The characteristic roots are at most triple, *i.e.*,

$$\partial_\tau^3 p(t, \tau, \xi) \ne 0 \quad \text{if } (t, \tau, \xi) \in [0, \infty) \times \mathbf{R} \times S^{n-1} \text{ and}$$
$$p(t, \tau, \xi) = \partial_\tau p(t, \tau, \xi) = \partial_\tau^2 p(t, \tau, \xi) = 0,$$

where $S^{n-1} = \{\xi \in \mathbf{R}^n;\ |\xi| = 1\}$. Let $\Gamma(p(t, \cdot, \cdot), \vartheta)$ be the connected component of the set $\{(\tau, \xi) \in \mathbf{R}^{n+1} \backslash \{0\};\ p(t, \tau, \xi) \ne 0\}$ which contains ϑ, and define the generalized flows $K^{\pm}_{(t_0, x^0)}$ for $p(t, \tau, \xi)$ by

$$\begin{aligned} K^{\pm}_{(t_0, x^0)} = \{&(t(s), x(s)) \in [0, \infty) \times \mathbf{R}^n;\ \pm s \ge 0 \text{ and } \{(t(s), x(s))\} \text{ is} \\ &\text{a Lipschitz continuous curve in } [0, \infty) \times \mathbf{R}^n \text{ satisfying} \\ &(d/ds)(t(s), x(s)) \in \Gamma(p(t(s), \cdot, \cdot), \vartheta)^* \ (a.e.\ s) \text{ and} \\ &(t(0), x(0)) = (t_0, x^0)\}, \end{aligned}$$

where $(t_0, x^0) \in [0, \infty) \times \mathbf{R}^n$ and $\Gamma^* = \{(t, x) \in \mathbf{R}^{n+1};\ t\tau + x \cdot \xi \geq 0$ for any $(\tau, \xi) \in \Gamma\}$. $K^+_{(t_0, x^0)}$ (resp. $K^-_{(t_0, x^0)}$) gives an estimate of the influence domain (resp. the dependence domain) of (t_0, x^0) (see Theorem 1.2 below). To describe conditions on the lower order terms we define the polynomials $h_j(t, \tau, \xi)$ $(\equiv h_j(t, \tau, \xi; p))$ of (τ, ξ) by

$$|p(t, \tau - i\gamma, \xi)|^2 = \sum_{j=0}^{m} \gamma^{2j} h_{m-j}(t, \tau, \xi) \quad \text{for } (t, \tau, \xi) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}^n \text{ and } \gamma \in \mathbf{R}.$$

Since $|p(t, \tau - i\gamma, \xi)|^2 = \prod_{j=1}^m ((\tau - \lambda_j(t, \xi))^2 + \gamma^2)$, we have

$$h_k(t, \tau, \xi) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq m} \prod_{l=1}^{k} (\tau - \lambda_{j_l}(t, \xi))^2 \quad (1 \leq k \leq m), \tag{1.1}$$

where $p(t, \tau, \xi) = \prod_{j=1}^m (\tau - \lambda_j(t, \xi))$. Let $\mathscr{R}(\xi)$ be a set-valued function, whose values are discrete subsets of $\mathbf{C}$, defined for $\xi \in S^{n-1}$ satisfying the following:

$$\begin{cases} \text{For any } T > 0 \text{ there is } N_T \in \mathbf{Z}_+ \text{ such that} \\ \#\{\lambda \in \mathscr{R}(\xi);\ \mathrm{Re}\,\lambda \in [0, T]\} \leq N_T \quad \text{for } \xi \in S^{n-1}. \end{cases}$$

Here $\#A$ denotes the number of the elements of a set A. We assume that $0 \in \mathscr{R}(\xi)$ when $\mathscr{R}(\xi) \neq \emptyset$. The subprincipal symbol of $P(t, D_t, D_x)$ is defined by

$$sub\ \sigma(P)(t, \tau, \xi) = P_{m-1}(t, \tau, \xi) + \frac{i}{2} \partial_t \partial_\tau p(t, \tau, \xi).$$

We assume

(L-1) for any $T > 0$ there is $C > 0$ such that

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t - s|, 1\right\} |sub\ \sigma(P)(t, \tau, \xi)| \tag{1.2}$$
$$\leq C h_{m-1}(t, \tau, \xi)^{1/2} \quad \text{for } (t, \tau, \xi) \in [0, T] \times \mathbf{R} \times S^{n-1}$$

as the Levi condition for the $(m-1)$-th order terms of P. Here we defined $\min_{s \in \mathscr{R}(\xi)} |t - s| = 1$ when $\mathscr{R}(\xi) = \emptyset$. To impose the Levi condition on the $(m-2)$-th order terms of P we have to define some quantities. Let $z^0 \equiv (t_0, \tau_0, \xi^0) \in [0, \infty) \times \mathbf{R} \times S^{n-1}$ satisfy $(\partial_\tau^k p)(z^0) = 0$ $(0 \leq k \leq 2)$. Define a monic polynomial $p(t, \tau, \xi; z^0)$ of τ of degree 3 satisfying the following:

$$\begin{cases} p(t,\tau,\xi;z^0) \text{ is defined for } (t,\xi)\in\mathscr{U}(z^0) \text{ and } p(t,\tau,\xi) \text{ is divided} \\ \text{by } p(t,\tau,\xi;z^0) \text{ as polynomials of } \tau, \text{ and, putting } \tilde{p}(t,\tau,\xi;z^0) = \\ p(t,\tau,\xi)/p(t,\tau,\xi;z^0), \\ \quad \tau\in I(z^0) \quad \text{if } (t,\xi)\in\mathscr{U}(z^0),\ |\xi|=1 \text{ and } p(t,\tau,\xi;z^0)=0, \\ \quad \tilde{p}(t,\tau,\xi;z^0)\neq 0 \quad \text{if } (t,\xi)\in\mathscr{U}(z^0),\ |\xi|=1 \text{ and } \tau\in I(z^0), \end{cases}$$

where $\mathscr{U}(z^0)$ is a neighborhood of (t_0,ξ^0) and $I(z^0)$ is a neighborhood of τ_0. Then we write

$$p(t,\tau,\xi;z^0) = \tau^3 + a_1(t,\xi;z^0)\tau^2 + a_2(t,\xi;z^0)\tau + a_3(t,\xi;z^0).$$

We define

$$\begin{aligned} (1.3)\qquad Q(t,\tau,\xi;z^0) = P_{m-2}(t,\tau,\xi) &+ \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{1}{4}\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\partial_t\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{i}{12}\partial_\tau^2\, sub\,\sigma(P)(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0) \\ &+ \frac{1}{24}(\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0))^2\cdot\partial_\tau\tilde{p}(t,\tau,\xi;z^0) \\ &\text{for } (t,\xi)\in\mathscr{U}(z^0) \text{ and } \tau\in\mathbf{R}. \end{aligned}$$

The Levi condition for the $(m-2)$-th order terms of P is the following:

(L-2) For any $z^0\in[0,\infty)\times\mathbf{R}\times S^{n-1}$ with $(\partial_\tau^k p)(z^0)=0$ $(0\le k\le 2)$ there is $C>0$ such that

$$\begin{aligned} (1.4)\qquad \min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|^2, 1\right\}&|Q(t,-a_1(t,\xi;z^0)/3,\xi;z^0)| \\ &\le Ch_{m-2}(t,-a_1(t,\xi;z^0)/3,\xi)^{1/2} \quad \text{for } (t,\xi)\in\mathscr{U}(z^0) \text{ with } |\xi|=1. \end{aligned}$$

We note that

$$(1.5)\quad Q(t,\tau,\xi;z^0) = P_1(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi) + \frac{i}{12}\partial_\tau^2 P_2(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi)$$

when $m=3$. In [5] we defined the sub-sub-principal symbol $sub^2\,\sigma(P)(t,\tau,\xi)$ of P by the right-hand side of (1.5).

Now we can state our main result.

THEOREM 1.2. *We assume that the conditions* (A-1), (A-2), (H), (T), (L-1) *and* (L-2) *are satisfied. Then the Cauchy problem* (CP) *is* C^∞ *well-posed. Moreover,* (CP) *has finite propagation property, more precisely, if* $(t_0, x^0) \in (0, \infty) \times \mathbf{R}^n$ *and* $u \in C^\infty([0, \infty) \times \mathbf{R}^n)$ *satisfies* (CP), $u_j(x) = 0$ *near* $\{x \in \mathbf{R}^n;\ (0, x) \in K^-_{(t_0, x^0)}\}$ $(0 \le j \le m-1)$ *and* $f = 0$ *near* $K^-_{(t_0, x^0)}$ (*in* $[0, \infty) \times \mathbf{R}^n$), *then* $(t_0, x^0) \notin \operatorname{supp} u$.

The remainder of this paper is organized as follows. We shall give preliminary lemmas in §2. In §3 we shall prove Theorem 1.2.

2. Preliminaries

Let I be an interval of $\mathbf{R}$, and let Γ be an open cone or a closed cone in $\mathbf{R}^n \backslash \{0\}$. Here 'cone' means that its vertex is the origin. Let $\kappa, \kappa' \in \mathbf{R}$. We say that $a(t, \xi) \in S^\kappa_{1,0}(I \times \Gamma)$ if $a(t, \xi) \in C^\infty(I \times \Gamma)$ and

$$(2.1) \qquad |D_t^j \partial_\xi^\alpha a(t, \xi)| \le C_{j,\alpha} |\xi|^{\kappa - |\alpha|}$$

$$\text{for } (t, \xi) \in I \times (\Gamma \cap \{|\xi| \ge 1\}) \text{ and any } j \in \mathbf{Z}_+ \text{ and } \alpha \in (\mathbf{Z}_+)^n.$$

When $a(t, \xi; \varepsilon)$ also depends on a parameter ε, we say that $a(t, \xi; \varepsilon) \in S^\kappa_{1,0}(I \times \Gamma)$ uniformly in ε if the $C_{j,\alpha}$ in (2.1) with $a(t, \xi)$ replaced by $a(t, \xi; \varepsilon)$ can be chosen so that they do not depend on ε. Moreover, we say that $a(t, \tau, \xi) \in \mathscr{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$ if $a(t, \tau, \xi) = \sum_{j=0}^{[\kappa]} a_j(t, \xi) \tau^j$ and $a_j(t, \xi) \in S^{\kappa+\kappa'-j}_{1,0}(I \times \Gamma)$, where $[\kappa]$ denotes the largest integer $\le \kappa$ and $\mathscr{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma) = \{0\}$ if $\kappa < 0$. We also write $\mathscr{S}^\kappa_{1,0}(I \times \Gamma) = \mathscr{S}^{\kappa,0}_{1,0}(I \times \Gamma)$ and $\mathscr{S}^{\kappa,-\infty}_{1,0}(I \times \Gamma) = \bigcap_{\kappa' \in \mathbf{R}} \mathscr{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$. When $a(t, \tau, \xi; \varepsilon) = \sum_{j=0}^{[\kappa]} a_j(t, \xi; \varepsilon) \tau^j$ depend on a parameter ε, we say that $a(t, \tau, \xi; \varepsilon) \in \mathscr{S}^{\kappa,\kappa'}_{1,0}(I \times \Gamma)$ uniformly in ε if $a_j(t, \xi; \varepsilon) \in S^{\kappa+\kappa'-j}_{1,0}(I \times \Gamma)$ uniformly in ε.

LEMMA 2.1. *Let* Γ *be a closed cone in* $\mathbf{R}^n \backslash \{0\}$, *and let* $a(t, \xi)$ *be a real analytic symbol defined in* $[0, 1] \times \Gamma$, *which is positively homogeneous in* ξ. *So there is a complex neighborhood* Ω *of* $[0, 1]$ *such that* $a(t, \xi)$ *is holomorphic in* $t \in \Omega$ *for* $\xi \in \Gamma$. *Put*

$$\mathscr{R}_a(\xi) = \begin{cases} \{\lambda \in \Omega;\ a(\lambda, \xi) = 0\} & \text{if } a(t, \xi) \not\equiv 0 \text{ in } t, \\ \varnothing & \text{if } a(t, \xi) \equiv 0 \text{ in } t \end{cases}$$

for $\xi \in \Gamma \cap S^{n-1}$. *Then there are* $N \in \mathbf{Z}_+$ *and* $C > 0$ *such that* $\# \mathscr{R}_a(\xi) \le N$ *for* $\xi \in \Gamma \cap S^{n-1}$ *and*

$$\min\Big\{\min_{s \in \mathscr{R}_a(\xi)} |t - s|, 1\Big\} |\partial_t a(t, \xi)| \le C |a(t, \xi)| \quad \text{for } (t, \xi) \in [0, 1] \times (\Gamma \cap S^{n-1}).$$

Remark. It follows from the proof that there are $C_k > 0$ ($k \in \mathbf{N}$) satisfying

$$\min\left\{\min_{s\in\mathscr{R}_a(\xi)} |t-s|^k, 1\right\}|\partial_t^k a(t,\xi)| \le C_k|a(t,\xi)| \quad \text{if } 1 \le k \le N$$

$$\min\left\{\min_{s\in\mathscr{R}_a(\xi)} |t-s|^N, 1\right\}|\partial_t^k a(t,\xi)| \le C_k|a(t,\xi)| \quad \text{if } k > N$$

for $(t,\xi) \in [0,1] \times (\Gamma \cap S^{n-1})$.

Proof. Replacing $[-\delta,\delta]$ and $\overline{U}$ with $[0,1]$ and $\{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2\}$, respectively, we apply the arguments as in the proof of Lemma 2.2 of [6]. Put

$$\kappa(\xi) = \int_0^1 |a(t,\xi)|^2\, dt$$

If $\kappa(\xi) \equiv 0$, then the lemma become trivial. So we may assume that $\kappa(\xi) \not\equiv 0$. Let $\xi^0 \in \{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2\}$. We apply Hironaka's resolution theorem to $\kappa(\xi)$ (see [1]). Then there are an open neighborhood $U(\xi^0)$ of ξ^0, a real analytic manifold $\tilde{U}(\xi^0)$, a proper analytic mapping $\varphi \equiv \varphi(\xi^0) : \tilde{U}(\xi^0) \ni \tilde{u} \mapsto \varphi(\tilde{u})$ $(\equiv \varphi(\tilde{u};\xi^0)) \in U(\xi^0)$ satisfying the following:

(i) $\varphi : \tilde{U}(\xi^0)\backslash\tilde{A} \to U(\xi^0)\backslash A$ is an isomorphism, where $A = \{\xi \in \Gamma;\ 1/2 \le |\xi| \le 2 \text{ and } \kappa(\xi) = 0\}$ and $\tilde{A} = \varphi^{-1}(A)$.

(ii) For each $p \in \tilde{U}(\xi^0)$ there are local analytic coordinates X $(\equiv X^p) = (X_1,\dots,X_n)$ $(= (X_1^p,\dots,X_n^p))$ centered at p, $r(p) \in \mathbf{Z}_+$ with $r(p) \le n$, $s_k(p) \in \mathbf{N}$ $(1 \le k \le r(p))$, a neighborhood $\tilde{U}(\xi^0;p)$ of p and a real analytic function $e(X)$ in $\tilde{V}(\xi^0;p)$ such that $e(X) > 0$ for $X \in \tilde{V}(\xi^0;p)$ and

$$\kappa(\varphi(\tilde{u})) = e(X(\tilde{u}))\prod_{k=1}^{r(p)} X_k(\tilde{u})^{2s_k(p)} \quad (\tilde{u} \in \tilde{U}(\xi^0;p)),$$

where $\tilde{V}(\xi^0;p) = \{X(\tilde{u});\ \tilde{u} \in \tilde{U}(\xi^0;p)\}$ and $\prod_{k=1}^{r(p)} \cdots = 1$ if $r(p) = 0$.

Here $\tilde{V}(\xi^0;p)$ is a neighborhood of 0 in $\mathbf{R}^n$ and we have used the fact that $\kappa(\xi) \ge 0$. Define $\tilde{\varphi}$ $(\equiv \tilde{\varphi}(\xi^0,p)) : \tilde{V}(\xi^0;p) \to U(\xi^0)$ by $\tilde{\varphi}(X(\tilde{u}))$ $(\equiv \tilde{\varphi}(X^p(\tilde{u});\xi^0,p)) = \varphi(\tilde{u})$ $(\equiv \varphi(\tilde{u};\xi^0))$ for $\tilde{u} \in \tilde{U}(\xi^0;p)$. Let $U_0(\xi^0)$ be a compact neighborhood of ξ^0 in $U(\xi^0)$, and put $\tilde{U}_0(\xi^0) = \varphi^{-1}(U_0(\xi^0))$. Fix $p \in \tilde{U}_0(\xi^0)$, and put

$$\alpha(p) = (s_1(p), \dots, s_{r(p)}(p), 0, \dots, 0) \in (\mathbf{Z}_+)^n,$$

$$c_\alpha(t; p) = \frac{1}{\alpha!} \partial_X^\alpha a(t, \tilde{\varphi}(X))|_{X=0}.$$

Note that $\alpha \geq \alpha(p)$ if $\alpha \in (\mathbf{Z}_+)^n$ and $c_\alpha(t; p) \not\equiv 0$ in t (see the proof of Lemma 2.2 of [6]). So we can write

$$a(t, \tilde{\varphi}(X)) = X^{\alpha(p)} a(t, X; p),$$

$$a(t, X; p) = c_{\alpha(p)}(t; p) + b(t, X; p),$$

where $b(t, X; p)$ is real analytic in (t, X) and satisfies $b(t, 0; p) = 0$. Since $c_{\alpha(p)}(t; p) \not\equiv 0$ in t, we can apply the Weierstrass preparation theorem to $a(t, X; p)$ at $(t, X) = (t_0, 0)$, where $t_0 \in [0, 1]$. Then there are $\delta(p, t_0) > 0$, a neighborhood $\tilde{V}(p, t_0)$ of 0 in $\tilde{V}(\xi^0; p)$, $m(p, t_0) \in \mathbf{Z}_+$, a real analytic function $c(t, X; p, t_0)$ defined in $[t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \tilde{V}(p, t_0)$ and real analytic functions $a_k(X; p, t_0)$ defined in $\tilde{V}(p, t_0)$ $(1 \leq k \leq m(p, t_0))$ such that $c(t, X; p, t_0) \neq 0$ and

$$a(t, X; p) = c(t, X; p, t_0)(t^{m(p, t_0)} + a_1(X; p, t_0) t^{m(p, t_0) - 1} + \cdots + a_{m(p, t_0)}(X; p, t_0))$$

in $[t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \tilde{V}(p, t_0)$. Note that $\delta(p, t_0)$, $\tilde{V}(p, t_0)$, $m(p, t_0)$, $c(t, X; p, t_0)$ and the $a_k(X; p, t_0)$ also depend on ξ^0. So we can write

$$a(t, \tilde{\varphi}(X)) = X^{\alpha(p)} c(t, X; p, t_0) \prod_{k=1}^{m(p, t_0)} (t - t_k(X; p, t_0))$$

for $(t, X) \in [t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \tilde{V}(p, t_0)$. We may assume that

$$\mathscr{R}_a(\tilde{\varphi}(X)) = \begin{cases} \{t_1(X; p, t_0), \dots, t_{m(p, t_0)}(X; p, t_0)\} & \text{if } X_1 \cdots X_{r(p)} \neq 0, \\ \varnothing & \text{if } X_1 \cdots X_{r(p)} = 0. \end{cases}$$

Then we have

$$\min\Big\{ \min_{s \in \mathscr{R}_a(\tilde{\varphi}(X))} |t - s|, 1 \Big\} |\partial_t a(t, \tilde{\varphi}(X))| \leq C(p, t_0) |a(t, \tilde{\varphi}(X))| \tag{2.2}$$

for $(t, X) \in [t_0 - \delta(p, t_0), t_0 + \delta(p, t_0)] \times \tilde{V}(p, t_0)$, where $C(p, t_0) > 0$. Since $[0, 1] \times \{\xi \in \Gamma;\ 1/2 \leq |\xi| \leq 2\}$ and $\tilde{U}_0(\xi^0)$ are compact, compactness arguments prove the lemma. □

From the assumption (T) there are $\delta_1 > 0$, $N_0 \in \mathbf{N}$, $m(j, k) \in \mathbf{N}$, open cones Γ_j in $\mathbf{R}^n \backslash \{0\}$, $r(j) \in \mathbf{N}$, compact intervals $J_{j,k}$ and $p^{j,k}(t, \tau, \xi) \in \mathscr{S}_{1,0}^{m(j,k)}([0, \delta_1] \times$

$(\bar{\Gamma}_j\backslash\{0\}))$ $(1 \le j \le N_0,\ 1 \le k \le r(j))$ such that $m(j,k) \le 3$, the $p^{j,k}(t,\tau,\xi)$ are monic polynomials of τ and positively homogeneous of degree $m(j,k)$ in $(\tau,\xi) \in \mathbf{R} \times (\bar{\Gamma}_j\backslash\{0\})$ such that $\bigcup_{j=1}^{N_0} \Gamma_j \supset S^{n-1}$, $J_{j,k} \cap J_{j,l} = \emptyset$ for $1 \le j \le N_0$ and $1 \le k < l \le r(j)$,

$$(2.3) \qquad p(t,\tau,\xi) = \prod_{k=1}^{r(j)} p^{j,k}(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma}_j \cap S^{n-1}),$$

$\tau \in J_{j,k}$ if $1 \le j \le N_0$, $1 \le k \le r(j)$, $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \cap S^{n-1})$, $\tau \in \mathbf{C}$ and $p^{j,k}(t,\tau,\xi) = 0$, and for each (j,k) with $1 \le j \le N_0$ and $1 \le k \le r(j)$ there is $(\hat{\tau},\hat{\xi}) \in \mathbf{R} \times (\Gamma_j \cap S^{n-1})$ satisfying

$$(\partial_\tau^\mu p^{j,k})(0,\hat{\tau},\hat{\xi}) = 0 \quad (0 \le \mu \le m(j,k) - 1).$$

Let $\delta > 0$ and Γ be a closed cone in $\mathbf{R}^n\backslash\{0\}$. We say that $a(t,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma)$ if $a(t,\xi)$ is real analytic in $[0,\delta] \times \Gamma$ and a classical symbol, *i.e.*, when $a(t,\xi) \not\equiv 0$, there are $\kappa \in \mathbf{Z}$ and real analytic symbols $a_j(t,\xi)$ $(j \in \mathbf{Z}_+)$ such that $a_0(t,\xi) \not\equiv 0$, $a_j(t,\xi)$ is positively homogeneous of degree $(\kappa - j)$ in ξ $(j \in \mathbf{Z}_+)$ and $a(t,\xi) \sim \sum_{j=0}^{\infty} a_j(t,\xi)$, *i.e.*,

$$\left| a(t,\xi) - \sum_{j=0}^{N-1} a_j(t,\xi) \right| \le C_N |\xi|^{\kappa - N}$$

for $(t,\xi) \in [0,\delta] \times \Gamma$ with $|\xi| \ge 1$ and $N \in \mathbf{N}$, where $C_N > 0$. Here $a_0(t,\xi)$ is called the principal symbol of $a(t,\xi)$.

Lemma 2.2. *Assume that $p(t,\tau,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma)[\tau]$ is a monic polynomial of τ, positively homogeneous of degree m $(\in \mathbf{N})$ in (τ,ξ) and hyperbolic in τ, i.e., $p(t,\tau \pm i,\xi) \ne 0$ for $(t,\tau,\xi) \in [0,\delta] \times \mathbf{R} \times \Gamma$. Write*

$$p(t,\tau,\xi) = \prod_{j=1}^{m} (\tau - \lambda_j(t,\xi)).$$

Then, for each fixed $\xi \in \Gamma \cap S^{n-1}$ we can enumerate $\{\lambda_j(t,\xi)\}$ so that the $\lambda_j(t,\xi)$ are real analytic in $t \in [0,\delta]$. Moreover, for any $\nu \in \mathbf{Z}_+$ there is $\mathscr{N}_\nu$ $(\equiv \mathscr{N}_\nu(p)) \subset \Gamma$ satisfying the following:

(i) *$\lambda\xi \in \mathscr{N}_\nu$ if $\lambda > 0$ and $\xi \in \mathscr{N}_\nu$.*
(ii) *$\mu_n(\mathscr{N}_\nu) = 0$.*
(iii) *There is $N_\nu \in \mathbf{Z}_+$ such that*

$\#\{t \in [0,\delta];\ \partial_t^\mu(\lambda_j(t,\xi) - \lambda_k(t,\xi)) = 0\} \le N_\nu$

if $0 \le \mu \le \nu,\ 1 \le j < k \le m$ *and* $\partial_t^\mu(\lambda_j(t,\xi) - \lambda_k(t,\xi)) \not\equiv 0$ *in* t,

$\#\{t \in [0,\delta];\ \partial_t^\mu \lambda_j(t,\xi) = 0\} \le N_\nu$

if $0 \le \mu \le \nu,\ 1 \le j \le m$ *and* $\partial_t^\mu \lambda_j(t,\xi) \not\equiv 0$ *in* t

for $\xi \in \Gamma \backslash \mathcal{N}_\nu$.

Here μ_n *denotes the Lebesgue measure in* $\mathbf{R}^n$.

Remark. The lemma is a generalization of Lemma 2.3 in [5]. We also need to apply the lemma to $\partial_\tau p^{j,k}(t,\tau,\xi)$ with $m(j,k) = 3$.

Proof. We will modify the proof of Lemma 2.3 of [5]. First fix $\xi \in \Gamma \cap S^{n-1}$. For $t_0 \in [0,\delta]$ $\mathscr{A}_{t_0}$ denotes the convergent power series ring of $(t - t_0)$. Since $\mathscr{A}_{t_0}$ is a unique factorization domain, $\mathscr{A}_{t_0}[\tau]$ is also a unique factorization domain. Applying the same argument as in the proof of Lemma 2.3 of [5], we can prove the first part of the lemma. In doing so we note that $\lambda_{j_0}(t_0 + z^r, \xi)$ is analytic in a complex neighborhood of $z = 0$ with some $r \in \mathbf{N}$ and that $\lambda_{j_0}(t_0 + z^r, \xi)$ can be expanded as a power series of z (see the proof of Lemma 2.3 of [5]). Hyperbolicity implies that $\lambda_{j_0}(t_0 + z^r, \xi)$ is real if z^r is real, and that $\lambda_{j_0}(t_0 + z^r, \xi)$ is a power series of z^r. So we can take $r = 1$ and $\lambda_{j_0}(t,\xi)$ is analytic in t near $t = t_0$. We denote by Σ the quotient field of $\mathscr{A}_{cl}([0,\delta] \times \Gamma)$. Then $\Sigma[\tau]$ is a unique factorization domain and $p(t,\tau,\xi) \in \Sigma[\tau]$. Write

$$\tau p(t,\tau,\xi) = p^1(t,\tau,\xi)^{r_1} \cdots p^\sigma(t,\tau,\xi)^{r_\sigma},$$

where $\sigma, r_j \in \mathbf{N}$, the $p^j(t,\tau,\xi)$ $(\in \Sigma[\tau])$ are irreducible in $\Sigma[\tau]$ and $p^j(t,\tau,\xi)$ and $p^k(t,\tau,\xi)$ are mutually prime if $j \ne k$. Define $q(t,\tau,\xi) = \prod_{j=1}^\sigma p^j(t,\tau,\xi)$, and let $D(t,\xi)$ be the discriminant of $q(t,\tau,\xi) = 0$ in τ. Then there are $d_k(t,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma) \backslash \{0\}$ $(k = 0,1)$ such that

$$D(t,\xi) = d_0(t,\xi)/d_1(t,\xi),$$

since $D(t,\xi) \ne 0$ in Σ. Here we may assume that the $d_k(t,\xi)$ are positively homogeneous in ξ (see the proof of Lemma 2.3 of [5]). Write $q(t,\tau,\xi) = \tau^{\hat{m}} + \sum_{j=1}^{\hat{m}-1} \hat{a}_j(t,\xi)\tau^{\hat{m}-j}$. Similarly, there are $\hat{a}_j^l(t,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma)$ $(1 \le j \le \hat{m} - 1,$ $l = 0,1)$ such that the $\hat{a}_j^l(t,\xi)$ are positively homogeneous in ξ, $\hat{a}_j^1(t,\xi) \ne 0$ (in $\mathscr{A}_{cl}([0,\delta] \times \Gamma)$) and $\hat{a}_j(t,\xi) = \hat{a}_j^0(t,\xi)/\hat{a}_j^1(t,\xi)$, since the $\hat{a}_j(t,\xi)$ are positively homogeneous in ξ. Put

$$\mathcal{N}_0 = \left\{ \xi \in \Gamma;\ d_0(t,\xi) d_1(t,\xi) \prod_{j=1}^{\hat{m}-1} \hat{a}_j^1(t,\xi) \equiv 0 \text{ in } t \in [0,\delta] \right\}.$$

Then we have $\mu_n(\mathcal{N}_0) = 0$. We can choose functions $\hat{\lambda}_j(t,\xi)$ $(1 \le j \le \hat{m})$ defined in $[0,\delta] \times (\Gamma \backslash \mathcal{N}_0)$ such that the $\hat{\lambda}_j(t,\xi)$ are real analytic in t for a fixed $\xi \in \Gamma \backslash \mathcal{N}_0$ and

$$q(t,\tau,\xi) = \prod_{j=1}^{\hat{m}} (\tau - \hat{\lambda}_j(t,\xi)) \quad \text{for } t \in [0,\delta] \text{ and } \xi \in \Gamma \backslash \mathcal{N}_0.$$

Note that

$$\{\hat{\lambda}_1(t,\xi), \dots, \hat{\lambda}_{\hat{m}}(t,\xi)\} = \{0, \lambda_1(t,\xi), \dots, \lambda_m(t,\xi)\}$$

for $(t,\xi) \in [0,\delta] \times (\Gamma \backslash \mathcal{N}_0)$. We may assume that $\hat{\lambda}_{\hat{m}}(t,\xi) \equiv 0$. Note that the $\hat{a}_j(t,\xi)$ are real analytic in $t \in [0,\delta]$ for $\xi \in \Gamma \backslash \mathcal{N}_0$. If $\xi \in \Gamma \backslash \mathcal{N}_0$ and

$$t \in D_\xi \equiv \left\{ s \in [0,\delta];\ d_0(s,\xi) d_1(s,\xi) \prod_{j=1}^{\hat{m}-1} \hat{a}_j^1(s,\xi) \neq 0 \right\},$$

then the roots of $q(t,\tau,\xi) = 0$ in τ are simple. It follows from Lemma 2.2 and its remark of [6] that there is $N_0 \in \mathbf{Z}_+$ satisfying $\#([0,\delta] \backslash D_\xi) \le N_0$ for $\xi \in \Gamma \backslash \mathcal{N}_0$. This proves the second part of the lemma for $\nu = 0$. Let $\xi \in \Gamma \backslash \mathcal{N}_0$. Then we have

$$\partial_\tau q(t,\tau,\xi)|_{\tau = \hat{\lambda}_j(t,\xi)} \cdot \partial_t \hat{\lambda}_j(t,\xi) + \partial_t q(t,\tau,\xi)|_{\tau = \hat{\lambda}_j(t,\xi)} = 0$$

for $1 \le j \le \hat{m}$. Therefore, for $t \in D_\xi$ we have

$$\partial_t \hat{\lambda}_j(t,\xi) = -\partial_t q(t,\tau,\xi)|_{\tau = \hat{\lambda}_j(t,\xi)} / \partial_\tau q(t,\tau,\xi)|_{\tau = \hat{\lambda}_j(t,\xi)} \quad (1 \le j \le \hat{m}).$$

Since $\hat{\lambda}_{\hat{m}}(t,\xi) \equiv 0$, we have

$$\prod_{j=1}^{\hat{m}} \partial_t \hat{\lambda}_j(t,\xi) \equiv 0.$$

Noting that

$$\prod_{k \neq j} (\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) \partial_\tau q(t,\tau,\xi)|_{\tau = \hat{\lambda}_k(t,\xi)}$$

$$= \prod_{1 \le k, l \le \hat{m}, k \neq l} (\hat{\lambda}_k(t,\xi) - \hat{\lambda}_l(t,\xi)) = (-1)^{\hat{m}(\hat{m}-1)/2} D(t,\xi)$$

for a fixed j with $1 \le j \le \hat{m}$, we can write the other fundamental symmetric expressions as follows:

$$\sum_{j=1}^{\hat{m}} \prod_{k \neq j} \partial_t \hat{\lambda}_k(t,\xi) = (-1)^{\hat{m}-1+\hat{m}(\hat{m}-1)/2}$$

$$\times \sum_{j=1}^{\hat{m}} \prod_{k \neq j} \{(\hat{\lambda}_k(t,\xi) - \hat{\lambda}_j(t,\xi))\partial_t q(t,\tau,\xi)|_{\tau=\hat{\lambda}_k(t,\xi)}\}/D(t,\xi)$$

$$= E_{\hat{m}-1}(t,\xi)/D(t,\xi),$$

$$\cdots$$

$$\sum_{j=1}^{\hat{m}} \partial_t \hat{\lambda}_j(t,\xi) = E_1(t,\xi)/D(t,\xi),$$

where the $E_k(t,\xi)$ are polynomials of $\{\partial_t^l \hat{a}_j(t,\xi)\}_{1 \le j \le \hat{m}-1, l=0,1}$. Put

$$\tilde{p}(t,\tau,\xi) = \tau^{\hat{m}} - E_1(t,\xi)D(t,\xi)^{-1}\tau^{\hat{m}-1} + E_2(t,\xi)D(t,\xi)^{-1}\tau^{\hat{m}-2}$$

$$+ \cdots + (-1)^{\hat{m}-1}E_{\hat{m}-1}(t,\xi)D(t,\xi)^{-1}\tau$$

$$\left(= \prod_{j=1}^{\hat{m}} (\tau - \partial_t \hat{\lambda}_j(t,\xi))\right).$$

Let us repeat the above argument with τp replaced by $\tilde{p}$. We write

$$\tilde{p}(t,\tau,\xi) = \tilde{p}^1(t,\tau,\xi)^{r_1'} \cdots \tilde{p}^{\sigma'}(t,\tau,\xi)^{r_{\sigma'}'},$$

where $\sigma', r_j' \in \mathbf{N}$, the $\tilde{p}^j(t,\tau,\xi)$ $(\in \Sigma[\tau])$ are irreducible in $\Sigma[\tau]$ and $\tilde{p}^j(t,\tau,\xi)$ and $\tilde{p}^k(t,\tau,\xi)$ are mutually prime if $j \neq k$. Put

$$\tilde{q}(t,\tau,\xi) = \prod_{j=1}^{\sigma'} \tilde{p}^j(t,\tau,\xi),$$

and let $\tilde{D}(t,\xi)$ be the discriminant of $\tilde{q}(t,\tau,\xi) = 0$ in τ. Then we can write

$$\tilde{D}(t,\xi) = \tilde{d}_0(t,\xi)/\tilde{d}_1(t,\xi),$$

where $\tilde{d}_k(t,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma) \setminus \{0\}$. Here we may assume that the $\tilde{d}_k(t,\xi)$ are positively homogeneous in ξ. Write

$$\tilde{q}(t,\tau,\xi) = \tau^{\tilde{m}} + \sum_{j=1}^{\tilde{m}-1} \tilde{a}_j(t,\xi)\tau^{\tilde{m}-j},$$

$$\tilde{a}_j(t,\xi) = \tilde{a}_j^0(t,\xi)/\tilde{a}_j^1(t,\xi) \quad (1 \le j \le \tilde{m}-1),$$

where $\tilde{a}_j^l(t,\xi) \in \mathscr{A}_{cl}([0,\delta] \times \Gamma)$ $(1 \le j \le \tilde{m}-1,\ l = 0,1)$ are positively homogeneous in ξ and $\tilde{a}_j^1(t,\xi) \ne 0$ in $\mathscr{A}_{cl}([0,\delta] \times \Gamma)$. Define

$$\mathscr{N}_1 = \left\{ \xi \in \Gamma;\ \tilde{d}_0(t,\xi)\tilde{d}_1(t,\xi) \prod_{j=1}^{\tilde{m}-1} \tilde{a}_j^1(t,\xi) \equiv 0 \ \text{ in } \ t \in [0,\delta] \right\} \cup \mathscr{N}_0.$$

Then we have $\mu_n(\mathscr{N}_1) = 0$. It is obvious that $\mathscr{N}_1$ is a cone. Similarly, there is $N_1 \in \mathbf{Z}_+$ such that

$$\#\{t \in [0,\delta];\ \partial_t(\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) = 0\}$$
$$\left(\le \# \left\{ t \in [0,\delta];\ \tilde{d}_0(t,\xi)\tilde{d}_1(t,\xi) \prod_{j=1}^{\tilde{m}-1} \tilde{a}_j^1(t,\xi) = 0 \right\} \right) \le N_1$$

if $\xi \in \Gamma \backslash \mathscr{N}_1$, $1 \le j < k \le \hat{m}$ and $\partial_t(\hat{\lambda}_j(t,\xi) - \hat{\lambda}_k(t,\xi)) \not\equiv 0$ in t. This proves the second part of the lemma for $v = 1$. Repeating the above arguments we can prove the lemma for $v = 2, 3, \dots,$ inductively. □

We choose $\rho(t) \in \mathscr{E}^{\{\kappa_0\}}(\mathbf{R})$ so that $\rho(t) \ge 0$, $\int \rho(t)\,dt = 1$ and $\operatorname{supp} \rho \subset \{t \in \mathbf{R};\ |t| \le 1\}$. Define

$$a_{j,\alpha}(t;\varepsilon) = \int \rho_\varepsilon(s) a_{j,\alpha}(t-s)\,ds \quad (3 \le j \le m,\ |\alpha| \le j-3),$$

$$P_k(t,\tau,\xi;\varepsilon) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t;\varepsilon)\tau^{m-j}\xi^\alpha \quad (0 \le k \le m-3),$$

$$P(t,\tau,\xi;\varepsilon) = \sum_{k=0}^{2} P_{m-k}(t,\tau,\xi) + \sum_{k=3}^{m} P_{m-k}(t,\tau,\xi;\varepsilon)$$

for $0 < \varepsilon \le 1$, where $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$.

We approximate $P(t,\tau,\xi)$ by $P(t,\tau,\xi;\varepsilon)$ in order to prove that (CP) has finite propagation property. We factorized $p(t,\tau,\xi)$ as (2.3). By the factorization theorem we can write

$$P(t,\tau,\xi;\varepsilon) = P^{j,1}(t,\tau,\xi;\varepsilon) \circ P^{j,2}(t,\tau,\xi;\varepsilon) \circ \cdots \circ P^{j,r(j)}(t,\tau,\xi;\varepsilon) + R_j(t,\tau,\xi;\varepsilon) \tag{2.4}$$

for $1 \le j \le N_0$, $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j$ with $|\xi| \ge 1$ and $\varepsilon \in (0,1]$, where

$$P^{j,k}(t,\tau,\xi;\varepsilon) = p^{j,k}(t,\tau,\xi) + q_0^{j,k}(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon),$$

$q_l^{j,k}(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(j,k)-1,-l}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ $(l = 0,1)$ are positively homogeneous of degree $(m(j,k)-1-l)$ in (τ,ξ) for $|\xi| \ge 1$, $r^{j,k}(t,\tau,\xi;\varepsilon) \in \mathscr{S}_{1,0}^{m(j,k)-1,-2}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ uniformly in ε and $R_j(t,\tau,\xi;\varepsilon) \in \mathscr{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ uniformly in ε (see, *e.g.*, [2] and, also, [6]). Here we denote by $a(t,\tau,\xi) \circ b(t,\tau,\xi)$ the symbol of $a(t,D_t,D_x)b(t,D_t,D_x)$. There are compact intervals $I_{j,k}$ $(1 \le j \le N_0,\ 1 \le k \le r(j))$ and $M > 0$ such that

$$\bigcup_{k=1}^{r(j)} I_{j,k} = [-M,M], \quad \mathring{I}_{j,k} \cap \mathring{I}_{j,l} = \varnothing \quad (1 \le j \le N_0,\ k \ne l),$$

$$\tau \in \mathring{I}_{j,k} \quad \text{if } 1 \le j \le N_0,\ 1 \le k \le r(j),\ (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma}_j \cap S^{n-1})$$
$$\text{and } p^{j,k}(t,\tau,\xi) = 0,$$

where $\mathring{I}$ denotes the interior of I $(\subset \mathbf{R})$. For $1 \le j \le N_0$ and $J \subset \{1,2,\dots,r(j)\}$ we define

$$\Pi_J^j(t,\tau,\xi) = \prod_{1 \le \mu \le r(j),\, \mu \notin J} p^{j,\mu}(t,\tau,\xi).$$

Now we fix j with $1 \le j \le N_0$. Until the end of the proof of Lemma 2.4 except the statements of Lemmas 2.3 and 2.4 we omit the subscript j and the superscript j of Γ_j, $P^{j,k}(\cdot)$, $R_j(\cdot)$, $p^{j,k}(\cdot)$, $I_{j,k}$, $\Pi_J^j(t,\tau,\xi), \dots$, and "$j$" of $r(j)$, $m(j,k), \dots$ and so on, *i.e.*, we write Γ_j, $P^{j,k}(\cdot)$, $R_j(\cdot)$, $p^{j,k}(\cdot)$, $I_{j,k}$, $\Pi_J^j(t,\tau,\xi)$, $r(j)$, $m(j,k), \dots$ as Γ, $P^k(\cdot)$, $R(\cdot)$, $p^k(\cdot)$, I_k, $\Pi_J(t,\tau,\xi)$, r, $m(k), \dots$, respectively. Let $a(t,\tau,\xi)$ and $b(t,\tau,\xi)$ be defined in $\mathscr{U}$. We write

$$a(t,\tau,\xi) = O(b(t,\tau,\xi)) \quad \text{for } (t,\tau,\xi) \in \mathscr{U}$$

if there is $C > 0$ satisfying

$$|a(t,\tau,\xi)| \le C|b(t,\tau,\xi)| \quad \text{for } (t,\tau,\xi) \in \mathscr{U}.$$

Assume that $a(t,\tau,\xi), b(t,\tau,\xi) \ge 0$. We write

$$a(t,\tau,\xi) \approx b(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in \mathscr{U}$$

if there is $C > 0$ satisfying

$$C^{-1}a(t,\tau,\xi) \le b(t,\tau,\xi) \le Ca(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in \mathscr{U}.$$

Lemma 2.3. *Let* $1 \le j \le N_0$. *We have*

$$(2.5)\quad sub\ \sigma(P)(t,\tau,\xi) = \sum_{k=1}^{r(j)} sub\ \sigma(P^{j,k})(t,\tau,\xi)\Pi^j_{\{k\}}(t,\tau,\xi) - \frac{i}{2}\sum_{1\le k<l\le r(j)} \{p^{j,k}, p^{j,l}\}(t,\tau,\xi)\Pi^j_{\{k,l\}}(t,\tau,\xi)$$

$$= \sum_{k=1}^{r(j)} sub\ \sigma(P^{j,k})(t,\tau,\xi)\Pi^j_{\{k\}}(t,\tau,\xi) + O(h_{m-1}(t,\tau,\xi)^{1/2})$$

$$\text{for } (t,\tau,\xi) \in [0,\delta_1]\times \mathbf{R} \times \bar{\Gamma}_j \text{ with } |\xi| \ge 1,$$

where

$$\{p^{j,k}, p^{j,l}\}(t,\tau,\xi) = \partial_\tau p^{j,k}(t,\tau,\xi)\cdot \partial_t p^{j,l}(t,\tau,\xi) - \partial_t p^{j,k}(t,\tau,\xi)\partial_\tau p^{j,l}(t,\tau,\xi).$$

Moreover, we have

$$(2.6)\quad P_{m-2}(t,\tau,\xi)$$

$$= \sum_{k=1}^{r(j)} q_1^{j,k}\Pi^j_{\{k\}} - i \sum_{1\le k<l\le r(j),\, \nu\ne k,l} \partial_\tau p^{j,k}\cdot \partial_t p^{j,l}\cdot sub\ \sigma(P^{j,\nu})\Pi^j_{\{k,l,\nu\}}$$

$$+ \sum_{1\le k<l\le r(j)} sub\ \sigma(P^{j,k})\ sub\ \sigma(P^{j,l})\Pi^j_{\{k,l\}}$$

$$- \frac{i}{2}\sum_{1\le k,l\le r(j),\, k\ne l} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,l})\Pi^j_{\{k,l\}}$$

$$- i \sum_{1\le k<l\le r(j)} \{\partial_\tau p^{j,k}\cdot \partial_t\ sub\ \sigma(P^{j,l}) + \partial_t p^{j,l}\cdot \partial_\tau\ sub\ \sigma(P^{j,k})\}\Pi^j_{\{k,l\}}$$

$$- \frac{1}{2}\sum_{1\le k<l\le r(j)} \{\partial_\tau p^{j,k}\cdot \partial_t^2\partial_\tau p^{j,l} + \partial_t p^{j,l}\cdot \partial_t\partial_\tau^2 p^{j,k}\}\Pi^j_{\{k,l\}}$$

$$+ O(h_{m-2}(t,\tau,\xi)^{1/2}) \quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times\bar{\Gamma}_j \text{ with } |\xi|\ge 1,$$

where $q^{j,k} = q^{j,k}(t,\tau,\xi)$, $\Pi^j_{\{k\}} = \Pi^j_{\{k\}}(t,\tau,\xi)$, $\partial_\tau p^{j,k} = \partial_\tau p^{j,k}(t,\tau,\xi)$, $sub\ \sigma(P^{j,\nu}) = sub\ \sigma(P^{j,\nu})(t,\tau,\xi),\ldots.$

Proof. We can prove by induction on r that

$$
\begin{aligned}
(2.7)\qquad & P^1(t,\tau,\xi;\varepsilon)\circ P^2(t,\tau,\xi;\varepsilon)\circ\cdots\circ P^r(t,\tau,\xi;\varepsilon) \\
& -\Bigg[\prod_{k=1}^r p^k(t,\tau,\xi)+\sum_{k=1}^r q_0^k(t,\tau,\xi)\Pi_{\{k\}} - i\sum_{1\le k<l\le r}\partial_\tau p^k\cdot\partial_t p^l\cdot\Pi_{\{k,l\}} \\
& \quad - i\sum_{1\le k<l\le r,\,\nu\ne k,l}\partial_\tau p^k\cdot\partial_t p^l\cdot q_0^\nu\Pi_{\{k,l,\nu\}}+\sum_{k=1}^r q_1^k\Pi_{\{k\}} \\
& \quad + \sum_{1\le k<l\le r} q_0^k q_0^l \Pi_{\{k,l\}} - i\sum_{1\le k<l\le r}\{\partial_\tau p^k\cdot\partial_t q_0^l+\partial_t p^l\cdot\partial_\tau q_0^k\}\Pi_{\{k,l\}} \\
& \quad - \frac{1}{2}\sum_{1\le k<l\le r}\partial_\tau^2 p^k\cdot\partial_t^2 p^l\cdot\Pi_{\{k,l\}} \\
& \quad - \sum_{1\le k<l<\nu\le r}\partial_\tau^2 p^k\cdot\partial_t p^l\cdot\partial_t p^\nu\cdot\Pi_{\{k,l,\nu\}} \\
& \quad - \sum_{1\le k<l<\nu\le r}\partial_\tau p^k\cdot\{\partial_t\partial_\tau p^l\cdot\partial_t p^\nu+\partial_\tau p^l\cdot\partial_t^2 p^\nu\}\Pi_{\{k,l,\nu\}} \\
& \quad - \sum_{\substack{1\le k<l\le r,\,k<\nu<\mu\le r\\ \nu\ne l,\,\mu\ne l}}\partial_\tau p^k\cdot\partial_t p^l\cdot\partial_\tau p^\nu\cdot\partial_t p^\mu\cdot\Pi_{\{k,l,\nu,\mu\}}\Bigg] \\
& \in \mathscr{S}_{1,0}^{m-1,-2}([0,\delta_1]\times(\bar{\Gamma}\setminus\{0\}))\quad \text{uniformly in } \varepsilon,
\end{aligned}
$$

where, for example, $\Pi_{\{k\}}=\prod_{1\le l\le r,\,l\ne k}p^l(t,\tau,\xi)$. It follows from (1.1) that

$$
\begin{aligned}
(2.8)\qquad & h_{m(k)-l}(t,\tau,\xi;p^k)\approx h_{m-l}(t,\tau,\xi) \\
& \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_k\times(\bar{\Gamma}\cap S^{n-1}) \\
& \text{if } 1\le k\le r \text{ and } 0\le l\le m(k),
\end{aligned}
$$

$$
\begin{aligned}
(2.9)\qquad & h_{m(k)-l}(t,\tau,\xi;p^k)\approx(|\tau|+1)^{2m(k)-2l} \\
& \text{for } (t,\tau,\xi)\in[0,\delta_1]\times(\mathbf{R}\setminus I_k)\times(\bar{\Gamma}\cap S^{n-1}) \\
& \text{if } 1\le k\le r \text{ and } 0\le l\le m(k),
\end{aligned}
$$

$$
\begin{aligned}
(2.10)\qquad & h_{m-l}(t,\tau,\xi)\approx(|\tau|+1)^{2m-2l} \\
& \text{for } (t,\tau,\xi)\in[0,\delta_1]\times((-\infty,-M)\cup(M,\infty))\times(\bar{\Gamma}\cap S^{n-1}) \\
& \text{if } 1\le l\le m.
\end{aligned}
$$

We have also, with $C > 0$,

$$|\partial_t^\mu \partial_\tau^\nu p^k(t,\tau,\xi)| \le C h_{m(k)-\mu-\nu}(t,\tau,\xi)^{1/2}(|\tau|+1)^\mu \tag{2.11}$$

for $1 \le k \le r$, $\mu, \nu \in \mathbf{Z}_+$ with $\mu + \nu < m(k)$ and $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma} \cap S^{n-1})$. (2.7)–(2.11) prove the lemma. □

Now assume that (L-1) is satisfied. Let $1 \le k_0 \le r$ with $m(k_0) = 3$. Then there is $C > 0$ such that

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t-s|, 1\right\} |sub\ \sigma(P^{k_0})(t,\tau,\xi)| \le C h_2(t,\tau,\xi;p^{k_0})^{1/2}$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma} \cap S^{n-1})$. Write

$$p^k(t,\tau,\xi) = \prod_{l=1}^{m(k)} (\tau - \lambda_l^k(t,\xi)),$$

$$p_\mu^k(t,\tau,\xi) = p^k(t,\tau,\xi)/(\tau - \lambda_\mu^k(t,\xi))$$

($1 \le k \le r$, $1 \le \mu \le m(k)$). Note that $h_2(t,\tau,\xi;p^{k_0}) = \sum_{\mu=1}^3 p_\mu^{k_0}(t,\tau,\xi)^2$. It follows from Lagrange's interpolation formula that there are functions $b_\mu(t,\xi)$ ($1 \le \mu \le 3$) and $C > 0$ satisfying

$$sub\ \sigma(P^{k_0})(t,\tau,\xi) = \sum_{\mu=1}^3 b_\mu(t,\xi) p_\mu^{k_0}(t,\tau,\xi), \tag{2.12}$$

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t-s|, 1\right\} |b_\mu(t,\xi)| \le C \tag{2.13}$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma} \cap S^{n-1})$ (see the proof of Lemma 2.5 of [5]).

Lemma 2.4. *Assume that* (L-1) *is satisfied, and that* $1 \le j \le N_0$, $1 \le k_0 \le r(j)$ *and* $m(j,k_0) = 3$. *Then there is* $C > 0$ *such that*

$$\begin{aligned} \min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\right\} &|\partial_\tau^\mu\ sub\ \sigma(P^{j,k_0})(t,\tau,\xi)| \\ &\le C h_{2-\mu}(t,\tau,\xi;p^{j,k_0})^{1/2} \quad (\mu \le 2), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|^2, 1\right\} &|\partial_t\ sub\ \sigma(P^{j,k_0})(t,\tau,\xi)|/(|\tau|+|\xi|) \\ &\le C h_1(t,\tau,\xi;p^{j,k_0})^{1/2} \end{aligned} \tag{2.15}$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \bar{\Gamma}_j$ *with* $|\xi| \ge 1$, *modifying* $\mathscr{R}(\xi)$ *if necessary.*

PROOF. (2.14) easily follows from (2.12) and (2.13). Write $p^{k_0}(t,\tau,\xi) = \tau^3 + a_1^{k_0}(t,\xi)\tau^2 + a_2^{k_0}(t,\xi)\tau + a_3^{k_0}(t,\xi)$. We have

$$\begin{aligned} sub\ \sigma(P^{k_0})(t,\tau,\xi) = {} & sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi) \\ & + (\tau + a_1^{k_0}(t,\xi)/3)(\partial_\tau\ sub\ \sigma(P^{k_0}))(t, -a_1^{k_0}(t,\xi)/3, \xi) \\ & + \frac{1}{2}(\tau + a_1^{k_0}(t,\xi)/3)^2(\partial_\tau^2\ sub\ \sigma(P^{k_0}))(t,0,\xi), \end{aligned}$$

noting that $\deg_\tau\ sub\ \sigma(P^{k_0})(t,\tau,\xi) \le 2$. Therefore, we have

$$\begin{aligned} (2.16) \qquad & |\partial_t\ sub\ \sigma(P^{k_0})(t,\tau,\xi)| \\ & \le |\partial_t\ sub\ \sigma(P^{k_0})(t, -a_1^{k_0}/3, \xi)| \\ & \quad + \frac{1}{3}|\partial_t a_1^{k_0}(t,\xi)| \cdot |(\partial_\tau\ sub\ \sigma(P^{k_0}))(t, -a_1^{k_0}/3, \xi)| \\ & \quad + |\tau + a_1^{k_0}/3| \cdot |\partial_t(\partial_\tau\ sub\ \sigma(P^{k_0}))(t, -a_1^{k_0}/3, \xi)| \\ & \quad + |\tau + a_1^{k_0}/3|\,|\partial_t a_1^{k_0}(t,\xi)|/3 \cdot |(\partial_\tau^2\ sub\ \sigma(P^{k_0}))(t,0,\xi)| \\ & \quad + \frac{1}{2}|\tau + a_1^{k_0}/3|^2 \cdot |\partial_t(\partial_\tau^2\ sub\ \sigma(P^{k_0}))(t,0,\xi)|. \end{aligned}$$

Modifying $\mathscr{R}(\xi)$ if necessary, we can assume that

$$\{\mathrm{Re}\,\lambda;\ \lambda \in \Omega_1 \text{ and } sub\ \sigma(P^{k_0})(\lambda, -a_1^{k_0}(\lambda,\xi)/3, \xi) = 0\} \subset \mathscr{R}(\xi)$$

$$\text{If } sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi) \not\equiv 0 \text{ in } t$$

for $\xi \in \bar{\Gamma} \cap S^{n-1}$, where Ω_1 is a compact complex neighborhood of $[0,\delta_1]$. Lemma 2.1 yields

$$\begin{aligned} & \min\Big\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|^2, 1\Big\}|\partial_t\ sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)| \\ & \quad \le C \min\Big\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\Big\}|sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)| \end{aligned}$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}$ with $|\xi| \ge 1$, where $C > 0$. Note that $-a_1^{k_0}(t,\xi)/3 \in I_{k_0}$ for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma} \cap S^{n-1})$. So by (L-1), (2.8) and (1.1) we have, with $C > 0$,

$$\begin{aligned} (2.17) \qquad & \min\Big\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|^2, 1\Big\}|\partial_t\ sub\ \sigma(P^{k_0})(t, -a_1^{k_0}(t,\xi)/3, \xi)|/|\xi| \\ & \le Ch_1(t, -a_1^{k_0}(t,\xi)/3, \xi; p^{k_0})^{1/2} \end{aligned}$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}$ with $|\xi| \geq 1$. Since

$$|\tau + a_1^{k_0}(t,\xi)/3| \leq \frac{1}{3}\sum_{\mu=1}^{3} |\tau - \lambda_\mu^{k_0}(t,\xi)| \leq h_1(t,\tau,\xi; p^{k_0})^{1/2},$$

$$\begin{aligned} h_1(t, -a_1^{k_0}(t,\xi)/3, \xi; p^{k_0})^{1/2} &\leq \sum_{\mu=1}^{3} |a_1^{k_0}(t,\xi)/3 + \lambda_\mu^{k_0}(t,\xi)| \\ &\leq \frac{2}{3}\{|\lambda_1^{k_0}(t,\xi) - \lambda_2^{k_0}(t,\xi)| + |\lambda_2^{k_0} - \lambda_3^{k_0}| + |\lambda_3^{k_0} - \lambda_1^{k_0}|\} \\ &\leq \frac{4}{3}\sum_{\mu=1}^{3} |\tau - \lambda_\mu^{k_0}(t,\xi)| \leq 4h_1(t,\tau,\xi; p^{k_0})^{1/2} \end{aligned}$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \bar{\Gamma}$ with $|\xi| \geq 1$, (2.14), (2.16) and (2.17) give (2.15). □

We wrote

$$p^{j,k}(t,\tau,\xi) = \tau^3 + a_1^{j,k}(t,\xi)\tau^2 + a_2^{j,k}(t,\xi)\tau + a_3^{j,k}(t,\xi)$$

if $1 \leq j \leq N_0$, $1 \leq k \leq r(j)$ and $m(j,k) = 3$. We say that (L-1) for $[0,\delta]$ is satisfied if (1.2) is satisfied with $[0,T]$ replaced by $[0,\delta]$, and that (L-2) for $[0,\delta]$ is satisfied if (1.4) is satisfied with $[0,\infty)$ replaced by $[0,\delta]$.

Lemma 2.5. (i) (L-1) *for* $[0,\delta_1]$ *is satisfied if and only if there is* $C > 0$ *such that*

$$\text{(2.18)} \quad \min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|, 1\right\} |sub\ \sigma(P^{j,k})(t,\tau,\xi)| \leq Ch_{m(j,k)-1}(t,\tau,\xi; p^{j,k})^{1/2} \quad \textit{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma} \cap S^{n-1})$$

provided $1 \leq j \leq N_0$, $1 \leq k \leq r(j)$ *and* $m(j,k) = 2$ *or* 3.

(ii) *Assume that* (L-1) *for* $[0,\delta_1]$ *is satisfied. Then* (L-2) *for* $[0,\delta_1]$ *is satisfied if and only if there is* $C > 0$ *such that*

$$\text{(2.19)} \quad \min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|^2, 1\right\} |sub^2\ \sigma(P^{j,k})(t, -a_1^{j,k}(t,\xi)/3, \xi)| \leq Ch_1(t, -a_1^{j,k}(t,\xi)/3, \xi; p^{j,k})^{1/2} \quad \textit{for } (t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \cap S^{n-1})$$

provided $1 \le j \le N_0,\ 1 \le k \le r(j)$ *and* $m(j,k) = 3$, *modifying* $\mathscr{R}(\xi)$ *if necessary, where*

$$sub^2\, \sigma(P^{j,k})(t,\tau,\xi) = q_1^{j,k}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi) + \frac{i}{12}\partial_\tau^2 q_0^{j,k}(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi). \tag{2.20}$$

REMARK. In the lemma the interval $[0,\delta_1]$ can be replaced by a closed subinterval of $[0,\delta_1]$. From (2.5) we can see that whether the *sub* $\sigma(P^{j,k})(t,\tau,\xi)$ satisfy (2.18) or not does not depend on the order of the product in (2.4) while they depend on the order. Moreover, (2.26) below implies that whether the $sub^2\, \sigma(P^{j,k})(t,\tau,\xi)$ satisfy (2.19) or not does not depend on the order of the product in (2.4) while they depend on the order.

PROOF. (2.5) and (2.8)–(2.10) prove the first assertion (i). Assume that $1 \le j \le N_0,\ z^0 = (t_0,\tau_0,\xi^0) \in [0,\delta_1]\times\mathbf{R}\times(\bar\Gamma_j\cap S^{n-1}),\ (\partial_\tau^l p)(z^0) = 0\ (0 \le l \le 2)$, $1 \le k_0 \le r(j),\ m(j,k_0) = 3$ and $\tau_0 \in \mathring{I}_{j,k_0}$. Note that $p(t,\tau,\xi;z^0) = p^{j,k_0}(t,\tau,\xi)$ and $\tilde p(t,\tau,\xi;z^0) = \Pi^j_{\{k_0\}}(t,\tau,\xi)$. Moreover, we may assume that $\mathscr{U}(z^0) = [0,\delta_1]\times(\bar\Gamma_j\backslash\{0\})$ and $I(z^0) = I_{j,k_0}$ in the definition of $Q(t,\tau,\xi;z^0)$, and $Q(t,\tau,\xi;z^0)$ is defined in $[0,\delta_1]\times\mathbf{R}\times(\bar\Gamma_j\backslash\{0\})$. We say that

$$a(t,\tau,\xi) \equiv 0 \quad (\mathrm{mod\ (L\text{-}2)})\ \text{at}\ z^0 \quad \text{for}\ (t,\xi)\in[0,\delta_1]\times\bar\Gamma_j\ \text{with}\ |\xi|\ge 1$$

if there is $C > 0$ such that

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|^2, 1\right\}a(t,-a_1^{j,k_0}(t,\xi)/3,\xi)| \le Ch_{m-2}(t,-a_1^{j,k_0}(t,\xi)/3,\xi)^{1/2} \quad \text{for}\ (t,\xi)\in[0,\delta_1]\times(\bar\Gamma_j\cap S^{n-1}).$$

(L-2) implies that $Q(t,\tau,\xi;z^0) \equiv 0$ (mod (L-2)) at z^0 for $(t,\xi)\in[0,\delta_1]\times\bar\Gamma_j$ with $|\xi|\ge 1$. It follows from (2.6) that

$$\begin{aligned} q_1^{j,k_0}(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) &= P_{m-2}(t,\tau,\xi) \\ &\quad + i\sum_{\substack{1\le k<l\le r(j)\\ k,l\ne k_0}}\partial_\tau p^{j,k}(t,\tau,\xi)\cdot\partial_t p^{j,l}(t,\tau,\xi)\cdot sub\ \sigma(P^{j,k_0})(t,\tau,\xi) \\ &\qquad \times \Pi^j_{\{k_0,k,l\}}(t,\tau,\xi) \end{aligned} \tag{2.21}$$

$$- \sum_{1 \le k \le r(j), k \ne k_0} sub\ \sigma(P^{j,k_0})\ sub\ \sigma(P^{j,k}) \Pi^j_{\{k_0,k\}}$$

$$+ \frac{i}{2} \sum_{1 \le k \le r(j), k \ne k_0} \partial_t \partial_\tau p^{j,k} \cdot sub\ \sigma(P^{j,k_0}) \Pi^j_{\{k_0,k\}}$$

$$+ i \sum_{k_0 < k \le r(j)} \partial_t p^{j,k} \cdot \partial_\tau\ sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}}$$

$$+ i \sum_{1 \le k < k_0} \partial_\tau p^{j,k} \cdot \partial_t\ sub\ \sigma(P^{j,k_0}) \cdot \Pi^j_{\{k_0,k\}}$$

$$+ \frac{1}{2} \sum_{k_0 < k \le r(j)} \partial_t p^{j,k} \cdot \partial_t \partial_\tau^2 p^{j,k_0} \cdot \Pi^j_{\{k_0,k\}}$$

$$+ \frac{1}{2} \sum_{1 \le k < k_0} \partial_\tau p^{j,k} \cdot \partial_t^2 \partial_\tau p^{j,k_0} \cdot \Pi^j_{\{k_0,k\}} + O(h_{m-2}(t,\tau,\xi)^{1/2})$$

for $(t,\tau,\xi) \in [0,\delta_1] \times I_{j,k_0} \times \bar{\Gamma}_j$ with $|\xi| \ge 1$,

where $sub\ \sigma(P^{j,k_0}) = sub\ \sigma(P^{j,k_0})(t,\tau,\xi), \ldots,$ since

$$\Pi^j_{\{k\}}(t,\tau,\xi) = O(p^{j,k_0}(t,\tau,\xi)|\xi|^{m-m(j,k)-3})$$

$$= O(h_{m-2}(t,\tau,\xi)^{1/2}|\xi|^{-m(j,k)+2}) \quad (k \ne k_0),$$

$$\partial_\tau p^{j,k_0}(t,\tau,\xi) = O(h_2(t,\tau,\xi; p^{j,k_0})^{1/2}),$$

$$\partial_t p^{j,k_0}(t,\tau,\xi) = O(h_2(t,\tau,\xi; p^{j,k_0})^{1/2}|\xi|),$$

$$\partial_t \partial_\tau p^{j,k_0}(t,\tau,\xi) = O(h_1(t,\tau,\xi; p^{j,k_0})^{1/2}|\xi|)$$

for $(t,\tau,\xi) \in [0,\delta_1] \times I_{j,k_0} \times \bar{\Gamma}$ with $|\xi| \ge 1$.

It also follows from (1.3) and Lemma 2.3 that

$$(2.22) \quad P_{m-2}(t,\tau,\xi) + \frac{1}{6} \partial_t^2 \partial_\tau^2 p^{j,k_0}(t,\tau,\xi) \cdot \Pi^j_{\{k_0\}}(t,\tau,\xi)$$

$$+ \frac{i}{12} \partial_\tau^2\ sub\ \sigma(P)(t,\tau,\xi) \cdot \partial_t \partial_\tau^2 p^{j,k_0}(t,\tau,\xi)$$

$$= Q(t,\tau,\xi; z^0) - \frac{1}{4} \partial_t \partial_\tau^2 p^{j,k_0}(t,\tau,\xi) \cdot \partial_t \Pi^j_{\{k_0\}}(t,\tau,\xi)$$

$$- \frac{1}{24} (\partial_t \partial_\tau^2 p^{j,k_0}(t,\tau,\xi))^2 \cdot \partial_\tau \Pi^j_{\{k_0\}}(t,\tau,\xi),$$

$$(2.23)\quad \{\partial_\tau^2 q_0^{j,k_0}(t,\tau,\xi)\cdot\Pi^j_{\{k_0\}}(t,\tau,\xi)-\partial_\tau^2\, sub\,\sigma(P)(t,\tau,\xi)\}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)$$
$$=-\{2\partial_\tau\, sub\,\sigma(P^{j,k_0})\cdot\partial_\tau\Pi^j_{\{k_0\}}(t,\tau,\xi)$$
$$+\,sub\,\sigma(P^{j,k_0})\cdot\partial_\tau^2\Pi^j_{\{k_0\}}(t,\tau,\xi)\}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)$$
$$+\frac{i}{2}\sum_{k_0<k\le r(j)}\{6\partial_t p^{j,k}-\partial_t\partial_\tau^2 p^{j,k_0}\cdot\partial_\tau p^{j,k}\}\Pi^j_{\{k_0,k\}}\cdot\partial_t\partial_\tau^2 p^{j,k_0}$$
$$-\frac{i}{2}\sum_{1\le k<k_0}\{6\partial_t p^{j,k}-\partial_t\partial_\tau^2 p^{j,k_0}\cdot\partial_\tau p^{j,k}\}\Pi^j_{\{k_0,k\}}\cdot\partial_t\partial_\tau^2 p^{j,k_0}$$
$$+\,O(h_{m-2}(t,\tau,\xi)^{1/2})\quad\text{for}\ \ (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\bar{\Gamma}_j\ \text{ with }\ |\xi|\ge 1,$$

since

$$\partial_\tau^2\, sub\,\sigma(P^{j,k_0})(t,\tau,\xi)=\partial_\tau^2 q_0^{j,k_0}(t,\tau,\xi),$$
$$\partial_\tau^2\{p^{j,k_0}(t,\tau,\xi),p^{j,k}(t,\tau,\xi)\}$$
$$=6\partial_t p^{j,k}(t,\tau,\xi)-\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot\partial_\tau p^{j,k}(t,\tau,\xi)$$
$$+\,O(h_1(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|^{m(j,k)-1})$$
$$\text{for}\ \ (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\bar{\Gamma}_j\ \text{ with }\ |\xi|\ge 1\ \text{ and }\ k\ne k_0,$$
$$h_1(t,\tau,\xi;p^{j,k_0})^{1/2}|\xi|^{m-3}=O(h_{m-2}(t,\tau,\xi)^{1/2})$$
$$\text{for}\ \ (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\bar{\Gamma}_j\ \text{ with }\ |\xi|\ge 1.$$

Therefore, (2.20)–(2.23) yield

$$(2.24)\quad sub^2\,\sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi)$$
$$=q_1^{j,k_0}(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi)$$
$$+\frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k_0}(t,\tau,\xi)\cdot\Pi^j_{\{k_0\}}+\frac{i}{12}\partial_\tau^2\, sub\,\sigma(P)\partial_t\partial_\tau^2 p^{j,k_0}$$
$$+\frac{i}{12}\{\partial_\tau^2 q_0^{j,k_0}\cdot\Pi^j_{\{k_0\}}-\partial_\tau^2\, sub\,\sigma(P)\}\partial_t\partial_\tau^2 p^{j,k_0}$$
$$=P_{m-2}(t,\tau,\xi)+\frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k_0}\cdot\Pi^j_{\{k_0\}}+\frac{i}{12}\partial_\tau^2\, sub\,\sigma(P)\cdot\partial_t\partial_\tau^2 p^{j,k_0}$$
$$+\,i\sum_{\substack{1\le k<l\le r(j)\\ k,l\ne k_0}}\partial_\tau p^{j,k}\cdot\partial_t p^{j,l}\cdot sub\,\sigma(P^{j,k_0})\Pi^j_{\{k_0,k,l\}}$$

$$
\begin{aligned}
&- \sum_{1\le k\le r(j),\,k\ne k_0} sub\ \sigma(P^{j,k_0})\ sub\ \sigma(P^{j,k})\Pi^j_{\{k_0,k\}}\\
&+\frac{i}{2}\sum_{1\le k\le r(j),\,k\ne k_0} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,k_0})\Pi^j_{\{k_0,k\}}\\
&+i\sum_{k_0<k\le r(j)} \partial_t p^{j,k}\cdot\partial_\tau\ sub\ \sigma(P^{j,k_0})\cdot\Pi^j_{\{k_0,k\}}\\
&+i\sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot\partial_t\ sub\ \sigma(P^{j,k_0})\cdot\Pi^j_{\{k_0,k\}}\\
&+\frac{1}{2}\sum_{k_0<k\le r(j)} \partial_t p^{j,k}\cdot\partial_t\partial_\tau^2 p^{j,k_0}\cdot\Pi^j_{\{k_0,k\}}\\
&+\frac{1}{2}\sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot\partial_t^2\partial_\tau p^{j,k_0}\cdot\Pi^j_{\{k_0,k\}}+O(h_{m-2}(t,\tau,\xi)^{1/2})\\
&+\frac{i}{12}\{\partial_\tau^2 q_0^{j,k_0}\cdot\Pi^j_{\{k_0\}}-\partial_\tau^2\ sub\ \sigma(P)\}\partial_t\partial_\tau^2 p^{j,k_0}\\
=\ & i\sum_{\substack{1\le k<l\le r(j)\\ k,l\ne k_0}} \partial_\tau p^{j,k}(t,\tau,\xi)\cdot\partial_t p^{j,l}(t,\tau,\xi)\cdot sub\ \sigma(P^{j,k_0})(t,\tau,\xi)\\
&\qquad\qquad\times\Pi^j_{\{k_0,k,l\}}(t,\tau,\xi)\\
&- \sum_{1\le k\le r(j),\,k\ne k_0} sub\ \sigma(P^{j,k_0})\ sub\ \sigma(P^{j,k})\Pi^j_{\{k_0,k\}}\\
&+\frac{i}{2}\sum_{1\le k\le r(j),\,k\ne k_0} \partial_t\partial_\tau p^{j,k}\cdot sub\ \sigma(P^{j,k_0})\Pi^j_{\{k_0,k\}}\\
&+i\sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot\partial_t\ sub\ \sigma(P^{j,k_0})\cdot\Pi^j_{\{k_0,k\}}\\
&+i\sum_{k_0<k\le r(j)} \partial_t p^{j,k}\cdot\partial_\tau\ sub\ \sigma(P^{j,k_0})\cdot\Pi^j_{\{k_0,k\}}\\
&+\frac{1}{2}\sum_{1\le k<k_0} \partial_\tau p^{j,k}\cdot\partial_t^2\partial_\tau p^{j,k_0}\cdot\Pi^j_{\{k_0,k\}}+Q(t,\tau,\xi;z^0)\\
&-\frac{1}{12}\sum_{1\le k<k_0} (\partial_t\partial_\tau^2 p^{j,k_0})^2\partial_\tau p^{j,k}\cdot\Pi^j_{\{k_0,k\}}
\end{aligned}
$$

$$-\frac{i}{12}\{2\partial_\tau\, sub\ \sigma(P^{j,k_0})\cdot\partial_\tau\Pi^j_{\{k_0\}} + sub\ \sigma(P^{j,k_0})\partial_\tau^2\Pi^j_{\{k_0\}}\}\partial_t\partial_\tau^2 p^{j,k_0}$$

$$+ O(h_{m-2}(t,\tau,\xi)^{1/2}) \quad \text{for } (t,\tau,\xi)\in[0,\delta_1]\times I_{j,k_0}\times\bar{\Gamma}_j \text{ with } |\xi|\geq 1,$$

since

$$\partial_\tau\Pi^j_{\{k_0\}}(t,\tau,\xi) = \sum_{k_0<k\leq r(j)}\partial_\tau p^{j,k}\cdot\Pi^j_{\{k_0,k\}} + \sum_{1\leq k<k_0}\partial_\tau p^{j,k}\cdot\Pi^j_{\{k_0,k\}},$$

$$\partial_t\Pi^j_{\{k_0\}}(t,\tau,\xi) = \sum_{k_0<k\leq r(j)}\partial_t p^{j,k}\cdot\Pi^j_{\{k_0,k\}} + \sum_{1\leq k<k_0}\partial_t p^{j,k}\cdot\Pi^j_{\{k_0,k\}}.$$

It follows from (2.14), (2.15), and (2.24) that

$$(2.25)\quad sub^2\ \sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi)$$

$$\equiv Q(t,\tau,\xi;z^0) + \frac{1}{2}\sum_{1\leq k<k_0}\left\{\partial_t^2\partial_\tau p^{j,k_0} - \frac{1}{6}(\partial_t\partial_\tau^2 p^{j,k_0})^2\right\}\partial_\tau p^{j,k}\cdot\Pi^j_{\{k_0,k\}}$$

$$(\text{mod (L-2)}) \text{ at } z^0 \quad \text{for } (t,\xi)\in[0,\delta_1]\times\bar{\Gamma}_j \text{ with } |\xi|\geq 1.$$

On the other hand, we have

$$(\partial_t^2\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi) - \frac{1}{6}((\partial_t\partial_\tau^2 p^{j,k_0})(t,-a_1^{j,k_0}/3,\xi))^2$$

$$= \partial_t\{(\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}/3,\xi)\}$$

since

$$\partial_t a_1^{j,k_0}(t,\xi) = \frac{1}{2}\partial_t\partial_\tau^2 p^{j,k_0}(t,\tau,\xi).$$

Modifying $\mathscr{R}(\xi)$ if necessary, we can assume that

$$\{\text{Re}\,\lambda;\ \lambda\in\Omega_1 \text{ and } (\partial_t\partial_\tau p^{j,k_0})(\lambda,-a_1^{j,k_0}(\lambda,\xi)/3,\xi)=0\}\subset\mathscr{R}(\xi)$$

$$\text{if } (\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi)\not\equiv 0 \text{ in } t,$$

where Ω_1 is a compact complex neighborhood of $[0,\delta_1]$. Since, with $C>0$,

$$|(\partial_t\partial_\tau p^{j,k_0})(t,-a_1^{j,k_0}(t,\xi)/3,\xi)|$$

$$\leq Ch_1(t,-a_1^{j,k_0}/3,\xi;p^{j,k_0})^{1/2}|\xi| \quad \text{for } (t,\xi)\in[0,\delta_1]\times\bar{\Gamma}_j \text{ with } |\xi|\geq 1,$$

Lemma 2.1 and (2.25) give

$$(2.26)\quad sub^2\,\sigma(P^{j,k_0})(t,\tau,\xi)\Pi^j_{\{k_0\}}(t,\tau,\xi) \equiv Q(t,\tau,\xi;z^0) \quad (\text{mod (L-2)}) \text{ at } z^0$$

$$\text{for } (t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j \text{ with } |\xi| \geq 1,$$

which proves the assertion (ii). □

3. Proof of Theorem 1.2

In this section we assume that the conditions (A-1), (A-2), (H) and (T) are satisfied. In order to prove Theorem 1.2 we first derive energy estimates for each factor in (2.4). Fix j with $1 \leq j \leq N_0$, and define

$$p^{(l)}(t,\tau,\xi) = \partial_\tau^l p(t,\tau,\xi) = \frac{m!}{(m-l)!}\prod_{\mu=1}^{m-l}(\tau - \lambda_\mu^{(l)}(t,\xi)) \quad (1 \leq l \leq m-1),$$

$$p^{j,k(l)}(t,\tau,\xi) = \partial_\tau^l p^{j,k}(t,\tau,\xi)$$

$$= \frac{m(j,k)!}{(m(j,k)-l)!}\prod_{\mu=1}^{m(j,k)-l}(\tau - \lambda_\mu^{j,k(l)}(t,\xi)) \quad (1 \leq l \leq m(j,k)-1)$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j$ with $|\xi| \geq 1$ and $1 \leq k \leq r(j)$, where $p(t,\tau,\xi) = \prod_{\mu=1}^m(\tau - \lambda_\mu(t,\xi))$ and $p^{j,k}(t,\tau,\xi) = \prod_{\mu=1}^{m(j,k)}(\tau - \lambda_\mu^{j,k}(t,\xi))$. Here, by Lemma 2.2 we may assume that the $\lambda_\mu(t,\xi)$, $\lambda_\mu^{(l)}(t,\xi)$, $\lambda_\mu^{j,k}(t,\xi)$ and $\lambda_\mu^{j,k(l)}(t,\xi)$ are real analytic in $t \in [0,\delta_1]$. We write, for $1 \leq k \leq r(j)$,

$$p_l^{j,k}(t,\tau,\xi) = \prod_{1 \leq \mu \leq m(j,k),\,\mu \neq l}(\tau - \lambda_\mu^{j,k}(t,\xi)) \quad \text{if } m(j,k) = 2 \text{ or } 3,$$

$$p_{i,l}^{j,k}(t,\tau,\xi) = \prod_{1 \leq \mu \leq m(j,k),\,\mu \neq i,l}(\tau - \lambda_\mu^{j,k}(t,\xi)) \quad \text{if } i \neq l \text{ and } m(j,k) = 3,$$

$$p_l^{j,k(1)}(t,\tau,\xi) = 3(\tau - \lambda_\mu^{j,k(1)}(t,\xi))$$

$$\text{if } m(j,k) = 3,\ l = 1,2 \text{ and } \{l,\mu\} = \{1,2\},$$

$$(3.1)\quad \mathscr{P}_l^{j,k}(t,\tau,\xi) = p_l^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi) \quad \text{if } m(j,k) = 2 \text{ or } 3.$$

Note that $\mathscr{P}_l^{j,k}(t,\tau,\xi) = p_l^{j,k}(t,\tau,\xi)$ if $m(j,k) = 2$.

Lemma 3.1. (i) (L-1) *for* $[0,\delta_1]$ *is satisfied if and only if there are symbols* $b_{1,l}^{j,k}(t,\xi)$ $(1 \leq l \leq m(j,k))$ *and* $C > 0$ *such that the* $b_{1,l}^{j,k}(t,\xi)$ *are positively* *homogeneous of degree* 0 *in* ξ *and*

$$\text{(3.2)} \qquad sub\ \sigma(P^{j,k})(t,\tau,\xi) = \sum_{l=1}^{m(j,k)} b_{1,l}^{j,k}(t,\xi) p_l^{j,k}(t,\tau,\xi),$$

$$\text{(3.3)} \qquad \min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|, 1\right\} |b_{1,l}^{j,k}(t,\xi)| \le C \quad (1 \le l \le m(j,k))$$

$$\textit{for } (t,\tau,\xi) \in ([0,\delta_1]\backslash\mathscr{R}(\xi)) \times \mathbf{R} \times (\bar{\Gamma}_j \cap S^{n-1})$$

provided that $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 2$ *or* 3.

(ii) *Assume that* (L-1) *for* $[0,\delta_1]$ *is satisfied. Then* (L-2) *for* $[0,\delta_1]$ *is satisfied if and only if there are symbols* $b_{2,l}^{j,k}(t,\xi)$ $(l = 1,2)$ *and* $C > 0$ *such that the* $b_{2,l}^{j,k}(t,\xi)$ *are positively homogeneous of degree* 0 *in* ξ *and*

$$sub^2\ \sigma(P^{j,k})(t,\tau,\xi) = \frac{1}{6}(\partial_\tau^2 q_1^{j,k})(t,0,\xi) \sum_{l=1}^{3} p_l^{j,k}(t,\tau,\xi)$$
$$+ \sum_{l=1}^{2} b_{2,l}^{j,k}(t,\xi) p_l^{j,k(1)}(t,\tau,\xi),$$

$$\text{(3.4)} \qquad \min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|^2, 1\right\} |b_{2,l}^{j,k}(t,\xi)| \le C \quad (l = 1,2)$$

$$\textit{for } (t,\tau,\xi) \in ([0,\delta_1]\backslash\mathscr{R}(\xi)) \times \mathbf{R} \times (\bar{\Gamma}_j \cap S^{n-1})$$

provided that $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 3$.

PROOF. Since

$$h_{m(j,k)-1}(t,\tau,\xi;p^{j,k}) = \sum_{l=1}^{m(j,k)} p_l^{j,k}(t,\tau,\xi)^2$$

$$\text{if } 1 \le j \le N_0,\ 1 \le k \le r(j) \text{ and } m(j,k) = 2 \text{ or } 3,$$

(2.18) and Lemma 2.5 of [5] with $r = m(j,k)$ prove the assertion (i). Let us prove the assertion (ii). Assume that $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 3$, and put

$$f(t,\tau,\xi) = sub^2\ \sigma(P^{j,k})(t,\tau,\xi) - \frac{1}{2}\partial_\tau^2 q_1^{j,k}(t,\tau,\xi)(\tau^2 - (a_1^{j,k}(t,\xi)/3)^2).$$

Note that $\partial_\tau^2 q_1^{j,k}(t,\tau,\xi)$ does not depend on τ. $f(t,\tau,\xi)$ is a polynomial of τ of degree 1 and positively homogeneous of degree 1 in (τ,ξ). Then we can prove that, with some $C_1, C_2 > 0$,

$$\min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|^2, 1\right\}|f(t,\tau,\xi)| \le C_1 h_1(t,\tau,\xi; p^{j,k})^{1/2} \tag{3.5}$$

$$\text{for } (t,\tau,\xi) \in [0,\delta_1]\times \mathbf{R}\times(\overline{\Gamma}_j \cap S^{n-1})$$

if and only if

$$\min\left\{\min_{s\in\mathscr{R}(\xi)} |t-s|^2, 1\right\}|f(t,-a_1^{j,k}(t,\xi)/3,\xi)|$$
$$\le C_2 h_1(t,-a_1^{j,k}(t,\xi)/3,\xi; p^{j,k})^{1/2} \quad \text{for } (t,\xi)\in[0,\delta_1]\times(\overline{\Gamma}_j\cap S^{n-1}).$$

Indeed, we have

$$f(t,\tau,\xi) = f(t,-a_1^{j,k}(t,\xi)/3,\xi) + (\tau + a_1^{j,k}(t,\xi)/3)\partial_\tau f(t,\tau,\xi)$$
$$= f(t,-a_1^{j,k}(t,\xi)/3,\xi) + O(h_1(t,\tau,\xi;p^{j,k})^{1/2}).$$

By (2.4) of [5] we have

$$h_1(t,\tau,\xi;p^{j,k}) \le \frac{1}{2}h_1(t,\tau,\xi;p^{j,k(1)}) = \frac{1}{2}\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)^2.$$

We have also

$$\tau^2 - (a_1^{j,k}(t,\xi)/3)^2$$
$$= \frac{1}{3}p^{j,k(1)}(t,\tau,\xi) - \frac{2}{9}a_1^{j,k}(t,\xi)\sum_{l=1}^{3}(\tau - \lambda_l^{j,k}(t,\xi))$$
$$+ \frac{\sqrt{3}}{18}(a_1^{j,k}(t,\xi)^2/3 - a_2^{j,k}(t,\xi))^{1/2}|p_1^{j,k(1)}(t,\tau,\xi) - p_2^{j,k(1)}(t,\tau,\xi)|,$$

$$\partial_\tau^2 q_1^k(t,\tau,\xi)(\tau^2 - (a_1^{j,k}(t,\xi)/3)^2) = \frac{1}{3}\partial_\tau^2 q_1^k(t,\tau,\xi)p^{j,k(1)}(t,\tau,\xi)$$
$$+ O(h_1(t,\tau,\xi;p^{j,k(1)})^{1/2}).$$

Since

$$f(t,-a_1^{j,k}(t,\xi)/3,\xi) = sub^2\,\sigma(P^{j,k})(t,-a_1^{j,k}(t,\xi)/3,\xi),$$

(2.19), (3.5) and Lemma 2.5 of [5] with $r = 2$ prove the assertion (ii). □

We assume that the hypotheses of Theorem 1.2 are fulfilled. Now let us repeat the same arguments as in §2 and §4 of [5]. Assume that $1 \le j \le N_0$ and $1 \le k \le r(j)$. It is easy to see that

$$\begin{aligned}
&(\tau - \lambda_l^{j,k}(t,\xi)) \circ \mathscr{P}_l^{j,k}(t,\tau,\xi) \\
&\quad = p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) \\
&\qquad - \frac{i}{2}\sum_{\mu \ne l} \partial_t(\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi)) \cdot p_{l,\mu}^{j,k}(t,\tau,\xi) - \frac{1}{2}\partial_t^2\partial_\tau p_l^{j,k}(t,\tau,\xi) \\
&\qquad \text{for } 1 \le l \le 3 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1
\end{aligned}$$

if $m(j,k) = 3$. So we have

$$\begin{aligned}
(3.6)\qquad &(\tau - \lambda_l^{j,k}(t,\xi)) \circ \mathscr{P}_l^{j,k}(t,\tau,\xi) \\
&\quad = p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) - \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi) \\
&\qquad - \frac{i}{2}\sum_{h \ne l} \partial_t(\lambda_l^{j,k}(t,\xi) - \lambda_h^{j,k}(t,\xi)) \cdot p_{l,h}^{j,k}(t,\tau,\xi) \\
&\qquad - \frac{1}{6}\partial_t^2\{(\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi)) + (\lambda_l^{j,k}(t,\xi) - \lambda_\nu^{j,k}(t,\xi))\} \\
&\qquad \text{for } 1 \le l \le 3 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1
\end{aligned}$$

if $m(j,k) = 3$, where $\{l,\mu,\nu\} = \{1,2,3\}$. We have also

$$\begin{aligned}
&(\tau - \lambda_l^{j,k}(t,\xi)) \circ p_l^{j,k}(t,\tau,\xi) \\
&\quad = p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t(\lambda_l^{j,k}(t,\xi) - \lambda_\mu^{j,k}(t,\xi)) \\
&\qquad \text{for } l = 1,2 \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j \text{ with } |\xi| \ge 1
\end{aligned}$$

if $m(j,k) = 2$, where $\{l,\mu\} = \{1,2\}$. Moreover, we have

$$\begin{aligned}
(3.7)\qquad &(\tau - \lambda_l^{j,k(1)}(t,\xi)) \circ p_l^{j,k(1)}(t,\tau,\xi) \\
&\quad = \sum_{\nu=1}^{3} \mathscr{P}_\nu^{j,k}(t,\tau,\xi) - \frac{3i}{2}\partial_t(\lambda_l^{j,k(1)}(t,\xi) - \lambda_\mu^{j,k(1)}(t,\xi))
\end{aligned}$$

if $m(j,k) = 3$, $l = 1,2$, $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times \overline{\Gamma}_j$, $|\xi| \ge 1$ and $\{l,\mu\} = \{1,2\}$.

(I) Let consider the case where $1 \le k \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 3$. Define

$$W_0^{j,k}(t,\xi;\gamma) = \sum_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,\delta_1+1]} \langle\xi\rangle_\gamma^{3/2}((t-s)^2\langle\xi\rangle_\gamma^{4/3}+1)^{-1/2},$$

$$+ \sum_{1\le l<\mu\le 3} \{(\partial_t(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi)))^2+1\}^{1/2}$$

$$\times \{(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))^2+1\}^{-1/2}+1,$$

$$W_1^{j,k}(t,\xi) = \sum_{1\le l<\mu\le 3} |\partial_t^2(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))|(|\partial_t(\lambda_l^{j,k}(t,\xi)-\lambda_\mu^{j,k}(t,\xi))|+1)^{-1}$$

$$+ |\partial_t(\lambda_2^{j,k(1)}(t,\xi)-\lambda_1^{j,k(1)}(t,\xi))|(|\lambda_2^{j,k(1)}(t,\xi)-\lambda_1^{j,k(1)}(t,\xi)|+1)^{-1},$$

$$\Lambda^{j,k}(t,\xi;\gamma) = \int_0^t (W_0^{j,k}(s,\xi;\gamma)+W_1^{j,k}(s,\xi))\,ds$$

for $(t,\xi) \in [0,\delta_1]\times(\bar{\Gamma}_j\backslash\mathscr{N}^{j,k})$ with $|\xi|\ge 1$ and $\gamma \ge 1$, where $\mathscr{N}^{j,k} = \mathscr{N}_2(p)\cup\mathscr{N}_1(p^{j,k(1)})\cup\{0\}$ and $\langle\xi\rangle_\gamma = (\gamma^2+|\xi|^2)^{1/2}$. It follows from Lemma 2.2, Lemma 2.4 of [5] and Theorem 1 of [4] that there is $C_0 > 0$ satisfying

$$0 \le \Lambda^{j,k}(t,\xi;\gamma) \le C_0(\log\langle\xi\rangle_\gamma + 1) \tag{3.8}$$

for $(t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}_j\backslash\mathscr{N}^{j,k})$ and $\gamma\ge 1$. For $(t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}_j\backslash\mathscr{N}^{j,k})$ with $|\xi|\ge 1$, $A\ge 1$ and $v(t,\xi)\in C^2([0,\delta_1];L^\infty(\mathbf{R}^n))$ we define

$$\mathscr{E}^{j,k}(t,\xi;v;\gamma,A) = \sum_{l=1}^3 e^{-A\Lambda^{j,k}}|\mathscr{P}_l^{j,k}v|^2 + \sum_{l=1}^2 W_0^{j,k}(t,\xi;\gamma)^2 e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2$$

$$+ W_0^{j,k}(t,\xi;\gamma)^4 e^{-A\Lambda^{j,k}}|v|^2,$$

where $\Lambda^{j,k} = \Lambda^{j,k}(t,\xi;\gamma)$, $\mathscr{P}_l^{j,k} = \mathscr{P}_l^{j,k}(t,D_t,\xi)$ and $p_l^{j,k(1)} = p_l^{j,k(1)}(t,D_t,\xi)$. Then we have

$$D_t\mathscr{E}^{j,k}(t,\xi;v;\gamma,A) \tag{3.9}$$

$$= i\sum_{l=1}^3 [A\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|\mathscr{P}_l^{j,k}v|^2 + 2\,\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t\mathscr{P}_l^{j,k}v)\cdot(\overline{\mathscr{P}_l^{j,k}v})\}]$$

$$+ i\sum_{l=1}^2 [(A(W_0^{j,k})^2\Lambda_t^{j,k} - 2W_0^{j,k}W_{0t}^{j,k})e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2$$

$$+ 2\,\mathrm{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}(D_t p_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\}]$$

$$+ i[(A(W_0^{j,k})^4\Lambda_t^{j,k} - 4(W_0^{j,k})^3 W_{0t}^{j,k})e^{-A\Lambda^{j,k}}|v|^2$$
$$+ 2\,\mathrm{Im}\{(W_0^{j,k})^4 e^{-A\Lambda^{j,k}}(D_t v)\cdot\bar{v}\}],$$

where $\Lambda_t^{j,k} = \partial_t\Lambda^{j,k}(t,\xi;\gamma)$ and $W_{0t}^{j,k} = \partial_t W_0^{j,k}(t,\xi;\gamma)$. Since the $\lambda_l^{j,k}(t,\xi)$ and the $\lambda_l^{j,k(1)}(t,\xi)$ are real-valued, it follows from (3.6) and (3.7) that

$$(3.10)\quad \mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t\mathscr{P}_l^{j,k}v)\cdot(\overline{\mathscr{P}_l^{j,k}v})\}$$
$$= \mathrm{Im}\{e^{-A\Lambda^{j,k}}((D_t-\lambda_l^{j,k})\mathscr{P}_l^{j,k}v)\cdot(\overline{\mathscr{P}_l^{j,k}v})\}$$
$$= \mathrm{Im}\left\{e^{-A\Lambda^{j,k}}\left(\left(p^{j,k}-\frac{i}{2}(\partial_t\partial_\tau p^{j,k})(t,D_t,\xi)\right)v\right)\cdot(\overline{\mathscr{P}_l^{j,k}v})\right\}$$
$$- \mathrm{Im}\{e^{-A\Lambda^{j,k}}((\partial_t^2\partial_\tau^2 p^{j,k})(t,D_t,\xi)v)\cdot(\overline{\mathscr{P}_l^{j,k}v})\}/6$$
$$- \mathrm{Re}\left\{e^{-A\Lambda^{j,k}}\sum_{\mu\neq l}(\lambda_{lt}^{j,k}-\lambda_{\mu t}^{j,k})(p_{l,\mu}^{j,k}v)\cdot(\overline{\mathscr{P}_l^{j,k}v})\right\}\Big/2$$
$$- \mathrm{Im}\left\{e^{-A\Lambda^{j,k}}\sum_{\mu\neq l}(\lambda_{ltt}^{j,k}-\lambda_{\mu tt}^{j,k})v\cdot(\overline{\mathscr{P}_l^{j,k}v})\right\}\Big/6,$$

$$(3.11)\quad \mathrm{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}(D_t p_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\}$$
$$= \mathrm{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}((D_t-\lambda_l^{j,k(1)})p_l^{j,k(1)}v)\cdot(\overline{p_l^{j,k(1)}v})\}$$
$$= \sum_{\mu=1}^{3}\mathrm{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}(\mathscr{P}_\mu^{j,k}v)\cdot(\overline{p_l^{j,k(1)}v})\}$$
$$- 3\,\mathrm{Re}\{(-1)^l(W_0^{j,k})^2 e^{-A\Lambda^{j,k}}(\lambda_{2t}^{j,k(1)}-\lambda_{1t}^{j,k(1)})v\cdot(\overline{p_l^{j,k(1)}v})\}/2,$$

$$(3.12)\quad \mathrm{Im}\{(W_0^{j,k})^4 e^{-A\Lambda^{j,k}}(D_t v)\cdot\bar{v}\} = \mathrm{Im}\left\{\sum_{l=1}^{2}(W_0^{j,k})^4 e^{-A\Lambda^{j,k}}(p_l^{j,k(1)}v)\cdot\bar{v}\right\}\Big/6,$$

where $\lambda_l^{j,k} = \lambda_l^{j,k}(t,\xi)$, $\lambda_{lt}^{j,k} = \partial_t\lambda_l^{j,k}(t,\xi)$, $\lambda_{ltt}^{j,k} = \partial_t^2\lambda_l^{j,k}(t,\xi)$, $p_{l,\mu}^{j,k} = p_{l,\mu}^{j,k}(t,D_t,\xi)$ and so forth. Put

$$\hat{f}_\varepsilon(t,\xi) = P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi),$$
$$q^{j,k}(t,\tau,\xi;\varepsilon) = \sum_{l=0}^{1} q_l^{j,k}(t,\tau,\xi) + r(t,\tau,\xi;\varepsilon)$$

and write

$$q_1^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon) = \sum_{l=0}^{2} \beta_l^{j,k}(t,\xi;\varepsilon)\tau^{2-l}$$

$$\beta_l^{j,k}(t,\xi;\varepsilon) = \beta_{l,0}^{j,k}(t,\xi) + \beta_{l,1}^{j,k}(t,\xi;\varepsilon),$$

where $\beta_{l,0}^{j,k}(t,\xi) \in S_{1,0}^{-1+l}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ is positively homogeneous of degree $(-1+l)$ in ξ and $\beta_{l,1}^{j,k}(t,\xi;\varepsilon) \in S_{1,0}^{-2+l}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ uniformly in ε ($l = 0, 1, 2$). Note that

$$q_1^{j,k}(t,\tau,\xi) = \sum_{l=0}^{2} \beta_{l,0}^{j,k}(t,\xi)\tau^{2-l}, \tag{3.13}$$

$$r^{j,k}(t,\tau,\xi;\varepsilon) = \sum_{l=0}^{2} \beta_{l,1}^{j,k}(t,\xi;\varepsilon)\tau^{2-l}.$$

Since

$$\begin{aligned} |\partial_t W_0^{j,k}(t,\xi;\gamma)| &\le W_0^{j,k}(t,\xi;\gamma)(W_0^{j,k}(t,\xi;\gamma) + \sqrt{2}W_1^{j,k}(t,\xi)) \\ &\le 2W_0^{j,k}(t,\xi;\gamma)\Lambda_t^{j,k}(t,\xi;\gamma), \end{aligned}$$

$$\begin{aligned} p^{j,k}(t,D_t,\xi)v(t,\xi) &= P^{j,k}(t,D_t,\xi;\varepsilon)v - q^{j,k}(t,D_t,\xi;\varepsilon)v \\ &= \hat{f}_\varepsilon(t,\xi) - q^{j,k}v, \end{aligned}$$

(3.9)–(3.12) yield

$$\begin{aligned} &\partial_t \mathscr{E}^{j,k}(t,\xi;v;\gamma,A) \\ &\quad \le 3(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|\hat{f}_\varepsilon(t,\xi)|^2 \\ &\qquad - \sum_{l=1}^{3}\Bigg[(A-4)\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|\mathscr{P}_l^{j,k}v|^2 \\ &\qquad\qquad - (\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\left|\left(q^{j,k} + \frac{i}{2}(\partial_t\partial_\tau p^{j,k}) + \frac{1}{6}(\partial_t^2\partial_\tau^2 p^{j,k})\right)v\right|^2 \\ &\qquad\qquad - (\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\sum_{\mu\ne l}|\lambda_{lt}^{j,k} - \lambda_{\mu t}^{j,k}|^2\{|p_{l,\mu}^{j,k}v|^2/4 + (W_1^{j,k})^2|v|^2/36\}\Bigg] \\ &\qquad - (A-10)\sum_{l=1}^{2}(W_0^{j,k})^2\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2 \end{aligned}$$

$$+ 2(W_0^{j,k})^2(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\left\{\sum_{l=1}^{3}|\mathscr{P}_l^{j,k}v|^2 + \frac{9}{4}(W_1^{j,k})^2|\lambda_2^{j,k(1)} - \lambda_1^{j,k(1)}|^2|v|^2\right\}$$

$$- (A-13)(W_0^{j,k})^4\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}|v|^2 + \sum_{l=1}^{2}(W_0^{j,k})^4(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}|p_l^{j,k(1)}v|^2/6,$$

where $(\partial_t\partial_\tau p^{j,k}) = (\partial_t\partial_\tau p^{j,k})(t, D_t, \xi), \dots$. First assume that $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^{j,k})$, $|\xi| \geq \gamma \geq 1$ and

$$\min\left\{\min_{s\in\mathscr{R}(\xi/|\xi|)}|t-s|, 1\right\} \leq \langle\xi\rangle_\gamma^{-2/3}. \tag{3.14}$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \geq \langle\xi\rangle_\gamma^{2/3}/\sqrt{2}.$$

So we have, with some $C > 0$,

$$(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\left|\left(q^{j,k} + \frac{i}{2}(\partial_t\partial_\tau p^{j,k}) + \frac{1}{6}(\partial_t^2\partial_\tau^2 p^{j,k})\right)v\right|^2$$

$$\leq C(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\left\{\sum_{l=1}^{3}|\mathscr{P}_l^{j,k}v|^2 + \sum_{l=1}^{2}(W_0^{j,k})^3|p_l^{j,k(1)}v|^2 + (W_0^{j,k})^6|v|^2\right\}$$

$$\leq C\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}\left\{\sum_{l=1}^{3}|\mathscr{P}_l^{j,k}v|^2 + \sum_{l=1}^{2}(W_0^{j,k})^2|p_l^{j,k(1)}v|^2 + (W_0^{j,k})^4|v|^2\right\},$$

since there are $c_\mu^{j,k}(t,\xi;\varepsilon) \in S_{1,0}^{\mu}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ $(\mu = 0,1,2)$ uniformly in ε satisfying

$$q^{j,k}(t,\tau,\xi;\varepsilon) + \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi)$$

$$= c_0^{j,k}(t,\xi;\varepsilon)\sum_{l=1}^{3}\mathscr{P}_l^{j,k}(t,\tau,\xi) + c_1^{j,k}(t,\xi;\varepsilon)\sum_{l=1}^{2}p_l^{j,k(1)}(t,\tau,\xi) + c_2^{j,k}(t,\xi;\varepsilon).$$

Note that there is $C > 0$ such that

$$|\lambda_{lt}^{j,k}(t,\xi)| \leq C|\xi| \quad \text{for } 1 \leq l \leq 3 \text{ and } (t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j \text{ with } |\xi| \geq 1$$

(see, *e.g.*, Theorem 1 of [4]). Then it follows from (4.11) of [5] that, with $C > 0$,

$$(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \sum_{\mu \neq l} |\lambda_{lt}^{j,k} - \lambda_{\mu t}^{j,k}|^2 \{|p_{l,\mu}^{j,k} v|^2/4 + (W_1^{j,k})^2 |v|^2/36\}$$

$$\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \left\{ \sum_{\mu=1}^{2} (W_0^{j,k})^2 |p_\mu^{j,k(1)} v|^2 + (W_0^{j,k})^4 |v|^2 \right\} \quad (1 \leq l \leq 3),$$

$$2(W_0^{j,k})^2 (\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \left\{ \sum_{\mu=1}^{3} |\mathscr{P}_\mu^{j,k} v| + 9(W_1^{j,k})^2 |\lambda_2^{j,k(1)} - \lambda_1^{j,k(1)}|^2 |v|^2/4 \right\}$$

$$\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \left\{ \sum_{\mu=1}^{3} |\mathscr{P}_\mu^{j,k} v|^2 + (W_0^{j,k})^2 \sum_{\mu=1}^{2} |p_\mu^{j,k(1)} v|^2 \right\}.$$

Therefore, there is $A_0 > 0$ satisfying

$$\partial_t \mathscr{E}^{j,k}(t, \xi; v; \gamma, A) \leq 3|\hat{f}_\varepsilon(t, \xi)|^2 \tag{3.15}$$

for $\varepsilon \in (0, 1]$ and $A \geq A_0$ if (3.14) is satisfied. Next assume that $(t, \xi) \in [0, \delta_1] \times (\overline{\Gamma}_j \backslash \mathscr{N}^{j,k})$, $|\xi| \geq \gamma \geq 1$ and (3.14) is not satisfied, *i.e.*,

$$\min\left\{ \min_{s \in \mathscr{R}(\xi/|\xi|)} |t - s|, 1 \right\} \geq \langle \xi \rangle_\gamma^{-2/3}.$$

Then we have

$$W_0^{j,k}(t, \xi; \gamma) \geq \left(\sqrt{2} \min\left\{ \min_{s \in \mathscr{R}(\xi/|\xi|) \cap [0, \delta_1 + 1]} |t - s|, 1 \right\} \right)^{-1}. \tag{3.16}$$

Operating ∂_τ^2 in the both sides of (3.2), we have

$$\partial_\tau^2 \, sub \, \sigma(P^{j,k})(t, \tau, \xi) = \partial_\tau^2 q_0^{j,k}(t, \tau, \xi) = 2 \sum_{l=1}^{3} b_{1,l}^{j,k}(t, \xi). \tag{3.17}$$

Since

$$\partial_t \partial_\tau p_l^{j,k}(t, \tau, \xi) = -\sum_{\mu \neq l} \lambda_{\mu t}^{j,k}(t, \xi),$$

$$\sum_{l=1}^{3} \partial_t \partial_\tau p_l^{j,k}(t, \tau, \xi) = \partial_t \partial_\tau^2 p^{j,k}(t, \tau, \xi) = 2\partial_t a_1^{j,k}(t, \xi) = -2 \sum_{l=1}^{3} \lambda_{lt}^{j,k}(t, \xi), \tag{3.18}$$

$$\partial_t \partial_\tau p_l^{j,k}(t, \tau, \xi) - \partial_t \partial_\tau^2 p^{j,k}(t, \tau, \xi)/3 = -\sum_{\mu \neq l} (\lambda_{\mu t}^{j,k}(t, \xi) - \lambda_{lt}^{j,k}(t, \xi))/3,$$

$$\tau = \sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/6 - a_1^{j,k}(t,\xi)/3,$$

$$\sum_{l=1}^{3} \mathscr{P}_l^{j,k}(t,\tau,\xi) = \partial_\tau p^{j,k}(t,\tau,\xi) - \frac{i}{2}\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi)$$

$$= 3\tau^2 + 2a_1^{j,k}(t,\xi)\tau + a_2^{j,k}(t,\xi) - i\partial_t a_1^{j,k}(t,\xi),$$

$$\tau^2 = \sum_{l=1}^{3} \mathscr{P}_l^{j,k}(t,\tau,\xi)/3 - a_1^{j,k}(t,\xi)\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/9$$

$$+ 2(a_1^{j,k}(t,\xi)/3)^2 + i\partial_t a_1^{j,k}(t,\xi)/3 - a_2^{j,k}(t,\xi)/3,$$

(2.20), (3.1), (3.17) and Lemma 3.1 give

$$q^{j,k}(t,\tau,\xi;\varepsilon) + \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi)$$

$$= sub\ \sigma(P^{j,k})(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon) + \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi)$$

$$= \sum_{l=1}^{3} b_{1,l}^{j,k}(t,\xi)\left\{\mathscr{P}_l^{j,k}(t,\tau,\xi) + \frac{i}{2}\left(\partial_t\partial_\tau p_l^{j,k}(t,\tau,\xi) - \frac{1}{3}\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi)\right)\right\}$$

$$+ \frac{i}{12}\partial_\tau^2 q_0^{j,k}(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p^{j,k}(t,\tau,\xi) + q_1^{j,k}(t,\tau,\xi)$$

$$+ \frac{1}{6}\partial_t^2\partial_\tau^2 p^{j,k}(t,\tau,\xi) + \beta_{0,1}^{j,k}(t,\xi;\varepsilon)\tau^2 + \beta_{1,1}^{j,k}(t,\xi;\varepsilon)\tau + \beta_{2,1}^{j,k}(t,\xi;\varepsilon)$$

$$= \sum_{l=1}^{3} b_{1,l}^{j,k}(t,\xi)\left\{\mathscr{P}_l^{j,k}(t,\xi) - \frac{i}{6}\sum_{\mu\neq l}(\lambda_{\mu t}^{j,k}(t,\xi) - \lambda_{lt}^{j,k}(t,\xi))\right\}$$

$$+ sub^2\ \sigma(P^{j,k})(t,\tau,\xi)$$

$$+ \beta_{0,1}^{j,k}(t,\xi;\varepsilon)\left\{\sum_{l=1}^{3}\mathscr{P}_l^{j,k}(t,\tau,\xi)/3 - a_1^{j,k}(\cdot)\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/9\right.$$

$$\left. + 2(a_1^{j,k}(\cdot)/3)^2 - a_2^{j,k}(\cdot)/3 + i\partial_t a_1^{j,k}(t,\xi)/3\right\}$$

$$+ \beta_{1,1}(t,\xi;\varepsilon)\left\{\sum_{l=1}^{2} p_l^{j,k(1)}(t,\tau,\xi)/6 - a_1^{j,k}(t,\xi)/3\right\} + \beta_{2,1}^{j,k}(t,\xi;\varepsilon)$$

$$= \sum_{l=1}^{3} (b_{1,l}^{j,k}(t,\xi) + \beta_0^{j,k}(t,\xi;\varepsilon)/3)\mathscr{P}_l^{j,k}(t,\tau,\xi)$$

$$- \frac{i}{6} \sum_{l=1}^{3} \sum_{\mu \neq l} b_{1,l}^{j,k}(t,\xi)(\lambda_{\mu t}^{j,k}(t,\xi) - \lambda_{lt}^{j,k}(t,\xi))$$

$$+ \sum_{l=1}^{2} \{b_{2,l}^{j,k}(t,\xi) + \beta_{1,1}^{j,k}(t,\xi;\varepsilon)/6 - \beta_{0,1}^{j,k}(t,\xi;\varepsilon)a_1^{j,k}(t,\xi)/9\} p_l^{j,k(1)}(t,\xi)$$

$$+ \gamma_0^{j,k}(t,\xi;\varepsilon),$$

where

$$\begin{aligned} \gamma_0^{j,k}(t,\xi;\varepsilon) = {} & \frac{i}{3}\beta_{0,0}^{j,k}(t,\xi)\partial_t a_1^{j,k}(t,\xi) \\ & + \beta_{0,1}^{j,k}(t,\xi;\varepsilon)(2(a_1^{j,k}(t,\xi)/3)^2 - a_2^{j,k}(t,\xi)/3 + i\partial_t a_1^{j,k}(t,\xi)/3) \\ & - \beta_{1,1}^{j,k}(t,\xi;\varepsilon)a_1^{j,k}(t,\xi)/3 + \beta_{2,1}^{j,k}(t,\xi;\varepsilon) \\ & \in S_{1,0}^0([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\})) \quad \text{uniformly in } \varepsilon \ (\in (0,1]). \end{aligned}$$

Here we have used the identity that

$$sub^2\, \sigma(P^{j,k})(t,\tau,\xi) = \frac{1}{3}\beta_{0,0}^{j,k}(t,\xi) \sum_{l=0}^{3} \mathscr{P}_l^{j,k}(t,\tau,\xi) + \frac{i}{3}\beta_{0,0}^{j,k}(t,\xi)\partial_t a_1^{j,k}(t,\xi)$$
$$+ \sum_{l=1}^{2} b_{2,l}^{j,k}(t,\xi) p_l^{j,k(1)}(t,\xi),$$

which follows from Lemma 3.1 (ii), (3.1), (3.13) and (3.18). By (3.3), (3.4), (3.16) and (4.11) of [5] there is $C > 0$ such that

$$(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} \left| \left(q^{j,k} + \frac{i}{2}(\partial_t\partial_\tau p^{j,k}) + \frac{1}{6}(\partial_t^2\partial_\tau^2 p^{j,k}) \right) v \right|^2$$
$$\leq C\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} \left\{ \sum_{l=1}^{3} |\mathscr{P}_l^{j,k} v|^2 + \sum_{l=1}^{2} (W_0^{j,k})^2 |p_l^{j,k(1)} v|^2 + (W_0^{j,k})^4 |v|^2 \right\}$$
$$\text{for } (t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^{j,k}) \text{ with } |\xi| \geq 1 \text{ and } \varepsilon \in (0,1],$$

since

$$|b_{v,l}^{j,k}(t,\xi)| \le CW_0^{j,k}(t,\xi;\gamma)^v \quad (v=1,2),$$

$$\begin{aligned}|(\lambda_{\mu t}^{j,k}(t,\xi)-\lambda_{lt}^{j,k}(t,\xi))v|^2 &\le W_0^{j,k}(t,\xi;\gamma)^2(|\lambda_\mu^{j,k}(t,\xi)-\lambda_l^{j,k}(t,\xi)|^2+1)|v|^2\\ &\le W_0^{j,k}(t,\xi;\gamma)^2(|(p_{\mu,v}^{j,k}(t,D_t,\xi)-p_{l,v}^{j,k}(t,D_t,\xi))v|^2+|v|^2)\\ &\le 2W_0^{j,k}(t,\xi;\gamma)^2\left(\sum_{h=1}^2 |p_h^{j,k(1)}v|^2+|v|^2\right),\end{aligned}$$

where $\{l,\mu,v\}=\{1,2,3\}$. Similarly, we have, with $C>0$,

$$\begin{aligned}&(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}\sum_{l=1}^3\sum_{\mu\ne l}|\lambda_{lt}^{j,k}-\lambda_{\mu t}^{j,k}|^2\{|p_{l,\mu}^{j,k}v|^2/4+(W_1^{j,k})^2|v|^2/36\}\\ &\quad\le \Lambda_t^{j,k}e^{-A\Lambda^{j,k}}\sum_{l=1}^3\sum_{\mu\ne l}(|\lambda_l^{j,k}-\lambda_\mu^{j,k}|^2+1)\{|p_{l,\mu}^{j,k}v|^2/4+(W_0^{j,k})^2|v|^2/36\}\\ &\quad\le C\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}\left\{\sum_{l=1}^3|\mathscr{P}_l^{j,k}v|^2+(W_0^{j,k})^2\sum_{l=1}^2|p_l^{j,k(1)}v|^2+(W_0^{j,k})^2|v|^2\right\},\\ &(W_0^{j,k})^2(\Lambda_t^{j,k})^{-1}e^{-A\Lambda^{j,k}}(W_1^{j,k})^2|\lambda_2^{j,k(1)}-\lambda_2^{j,k(1)}|^2|v|^2\\ &\quad\le 2\Lambda_t^{j,k}e^{-A\Lambda^{j,k}}(W_0^{j,k})^2\sum_{l=1}^2|p_l^{j,k(1)}v|^2/9,\end{aligned}$$

since

$$\begin{aligned}&|(\lambda_l^{j,k}-\lambda_\mu^{j,k})p_{l,\mu}^{j,k}v|^2\\ &\quad=\left|\mathscr{P}_l^{j,k}v-\mathscr{P}_\mu^{j,k}v+\frac{i}{2}(\lambda_{lt}^{j,k}-\lambda_{\mu t}^{j,k})v\right|^2\\ &\quad\le 3\left\{|\mathscr{P}_l^{j,k}v|^2+|\mathscr{P}_\mu^{j,k}v|^2+2(W_0^{j,k})^2\left(\sum_{h=1}^2|p_h^{j,k(1)}v|^2/9+|v|^2\right)\right\},\\ &(\lambda_2^{j,k(1)}-\lambda_1^{j,k(1)})v=(p_2^{j,k(1)}-p_1^{j,k(1)})v/3.\end{aligned}$$

Therefore, modifying A_0 if necessary, we can see that (3.15) holds for $(t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}_j\backslash\mathscr{N}^{j,k})$ with $|\xi|\ge 1$, $\varepsilon\in(0,1]$ and $A\ge A_0$. This gives

$$\mathscr{E}^{j,k}(t,\xi;v;\gamma,A)\le\mathscr{E}^{j,k}(0,\xi;v;\gamma,A)+3\int_0^t|\hat{f}_\varepsilon(s,\xi)|^2\,ds \tag{3.19}$$

if $A \ge A_0$, $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^{j,k})$, $|\xi| \ge \gamma \ge 1$ and $\varepsilon \in (0,1]$. We note that A_0 and C_0 in (3.8) depend on $P^{j,k}(t,\tau,\xi;\varepsilon)$.

Lemma 3.2. *Assume that* $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 3$. *Then there are* $c > 0$ *and* $C_A > 0$ *such that*

$$c\mathscr{E}^{j,k}(t,\xi;v;\gamma,A) \le \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{4-2l} |D_t^l v(t,\xi)|^2 \le C_A \langle\xi\rangle_\gamma^{4+AC_0} \mathscr{E}^{j,k}(t,\xi;v;\gamma,A)$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^{j,k})$ *with* $|\xi| \ge \gamma \ge 1$, $\varepsilon \in (0,1]$ *and* $v \in C^2([0,\delta_1]; L^\infty(\mathbf{R}^n))$.

Proof. We can write

$$D_t^2 v(t,\xi) = \frac{1}{3}\sum_{l=1}^{3} \mathscr{P}_l^{j,k}(t,D_t,\xi)v(t,\xi) + \sum_{l=0}^{1} c_l(t,\xi) D_t^l v(t,\xi) + \frac{i}{3}\partial_t a_1^{j,k}(t,\xi)\cdot v(t,\xi),$$

where $|c_l(t,\xi)| \le C|\xi|^{2-l}$. Similarly, we have

$$D_t v(t,\xi) = \frac{1}{6}\sum_{l=1}^{2} p_l^{j,k(1)}(t,D_t,\xi)v(t,\xi) + d^{j,k}(t,\xi)v(t,\xi),$$

where $|d^{j,k}(t,\xi)| \le C|\xi|$. Therefore, we have

$$\begin{aligned}
&\sum_{l=0}^{2} \langle\xi\rangle_\gamma^{4-2l} |D_t^l v(t,\xi)|^2 \\
&\quad \le C_A \langle\xi\rangle_\gamma^{4+AC_0} e^{-A\Lambda^{j,k}} \left\{ \sum_{l=1}^{3} |\mathscr{P}_l^{j,k} v|^2 + \sum_{l=1}^{2} (W_0^{j,k})^2 |p_l^{j,k(1)} v|^2 + (W_0^{j,k})^4 |v|^2 \right\} \\
&\quad \le C_A \langle\xi\rangle_\gamma^{4+AC_0} \mathscr{E}^{j,k}(t,\xi;v;\gamma,A).
\end{aligned}$$

It is obvious that, with $C > 0$,

$$\mathscr{E}^{j,k}(t,\xi;v;\gamma,A) \le C\sum_{l=0}^{2} \langle\xi\rangle_\gamma^{4-2l} |D_t^l v(t,\xi)|^2,$$

since $W_0^{j,k}(t,\xi;\gamma) \le C\langle\xi\rangle_\gamma^{2/3}$. □

Lemma 3.3. *Assume that* $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 3$. *Then for* $\mu \in \mathbf{N}$ *with* $\mu \ge 2$ *and* $\kappa \in \mathbf{R}$ *there are* $\nu_{j,k} > 0$ *and* $C_\mu > 0$ *such that*

$$(3.20)\qquad \sum_{l=0}^{\mu} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_t^l v(t,\xi)|^2$$

$$\leq C_\mu \left\{ \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{2\mu+2\kappa+4+\nu_{j,k}-2l} |(D_t^l v)(0,\xi)|^2 \right.$$

$$+ \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa+\nu_{j,k}} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds$$

$$\left. + \sum_{l=0}^{\mu-3} \langle\xi\rangle_\gamma^{2\mu+2\kappa-6-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \right\}$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^{j,k})$ *with* $|\xi| \geq \gamma \geq 1$, $\varepsilon \in (0,1]$ *and* $v \in C^\infty([0,\delta_1]; L^\infty(\mathbf{R}^n))$, *where* $\sum_{l=0}^{\mu-3} \cdots = 0$ *when* $\mu = 2$ *and the* $\nu_{j,k}$ *do not depend on* μ.

PROOF. From (3.19) with $A = A_0$ and Lemma 3.2 with $A = A_0$ it follows that (3.20) is valid for $\mu = 2$ if $\nu_{j,k} \geq A_0 C_0$. Let $M \geq 2$, and assume that (3.20) is valid for $\mu = M$. Then we have

$$(3.21)\qquad \sum_{l=0}^{M+1} \langle\xi\rangle_\gamma^{2M+2+2\kappa-2l} |D_t^l v(t,\xi)|^2$$

$$\leq C_M \left\{ \sum_{l=0}^{2} \langle\xi\rangle_\gamma^{2M+2+2\kappa+4+\nu_{j,k}-2l} |(D_t^l v)(0,\xi)|^2 \right.$$

$$+ \int_0^t \langle\xi\rangle_\gamma^{2M+2+2\kappa+\nu_{j,k}} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds$$

$$\left. + \sum_{l=0}^{\mu-3} \langle\xi\rangle_\gamma^{2M+2+2\kappa-6-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \right\}$$

$$+ \langle\xi\rangle_\gamma^{2\kappa} |D_t^{M+1} v(t,\xi)|^2$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^{j,k})$ with $|\xi| \geq \gamma \geq 1$, $\varepsilon \in (0,1]$ and $v \in C^\infty([0,\delta_1]; L^\infty(\mathbf{R}^n))$. On the other hand, we have

$$D_t^3 v(t,\xi) = -\sum_{l=0}^{2} a_{3-l}^{j,k}(t,\xi;\varepsilon) D_t^l v(t,\xi) + P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi),$$

where $P^{j,k}(t,\tau,\xi;\varepsilon) = \tau^3 + \sum_{l=1}^3 a_l^{j,k}(t,\xi;\varepsilon)\tau^{3-l}$. By induction we can easily show that for $h \in \mathbf{Z}_+$ there are symbols $a_{3+h-l}^{j,k,h}(t,\xi;\varepsilon) \in S_{1,0}^{3+h-l}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$

$(l = 0, 1, 2)$ and $b_{h-l}^{j,k,h}(t, \xi; \varepsilon) \in S_{1,0}^{h-l}([0, \delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ uniformly in $\varepsilon \in (0, 1]$ $(0 \le l \le h)$ satisfying

$$D_t^{3+h} v(t, \xi) = \sum_{l=0}^{2} a_{3+h-l}^{j,k,h}(t, \xi; \varepsilon) D_t^l v(t, \xi) + \sum_{l=0}^{h} b_{h-l}^{j,k,h}(t, \xi; \varepsilon) D_t^l P^{j,k}(t, D_t, \xi; \varepsilon) v(t, \xi).$$

This, with (3.20) for $\mu = 2$ and (3.21), proves that (3.20) is valid for $\mu = M + 1$. □

(II) Next consider the case where $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j, k) = 2$. Define

$$W_0^{j,k}(t, \xi; \gamma) = \sum_{s \in \mathscr{R}(\xi/|\xi|) \cap [0, \delta_1]} \langle \xi \rangle_\gamma^{1/2} ((t - s)^2 \langle \xi \rangle_\gamma + 1)^{-1/2} + 1,$$

$$W_1^{j,k}(t, \xi) = |\partial_t (\lambda_1^{j,k}(t, \xi) - \lambda_2^{j,k}(t, \xi))| / (|\lambda_1^{j,k}(t, \xi) - \lambda_2^{j,k}(t, \xi)| + 1),$$

$$\Lambda^{j,k}(t, \xi; \gamma) = \int_0^t (W_0(s, \xi; \gamma) + W_1(s, \xi))\, ds$$

for $(t, \xi) \in [0, \delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^0)$ with $|\xi| \ge 1$ and $\gamma \ge 1$, where $\mathscr{N}^0 = \mathscr{N}_2(p) \cup \{0\}$. Similarly, we have

$$|\partial_t W_0^{j,k}(t, \xi; \gamma)| \le W_0^{j,k}(t, \xi; \gamma)^2,$$

$$0 \le \Lambda^{j,k}(t, \xi; \gamma) \le C_0 (\log \langle \xi \rangle_\gamma + 1)$$

for $(t, \xi) \in [0, \delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^0)$ with $|\xi| \ge 1$, where $C_0 > 0$. For $(t, \xi) \in [0, \delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^0)$ with $|\xi| \ge 1$, $A \ge 1$ and $v(t, \xi) \in C^1([0, \delta_1]; L^\infty(\mathbf{R}^n))$ we define

$$\mathscr{E}^{j,k}(t, \xi; v; \gamma, A) = \sum_{l=1}^{2} e^{-A\Lambda^{j,k}} |p_l^{j,k} v|^2 + W_0^{j,k}(t, \xi; \gamma)^2 e^{-A\Lambda^{j,k}} |v|^2,$$

where $\Lambda^{j,k} = \Lambda^{j,k}(t, \xi; \gamma)$ and $p_l^{j,k} = p_l^{j,k}(t, D_t, \xi)$. Then we have

$$\begin{aligned} D_t \mathscr{E}^{j,k}(t, \xi; v; \gamma, A) = {} & i \sum_{l=1}^{2} A \Lambda_t^{j,k} e^{-A\Lambda^{j,k}} |p_l^{j,k} v|^2 + 2i \operatorname{Im}\{e^{-A\Lambda^{j,k}} (D_t p_l^{j,k} v) \cdot (\overline{p_l^{j,k} v})\} \\ & + i(A \Lambda_t^{j,k} (W_0^{j,k})^2 - 2 W_0^{j,k} W_{0t}^{j,k}) e^{-A\Lambda^{j,k}} |v|^2 \\ & + 2i \operatorname{Im}\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}} (D_t v) \cdot \bar{v}\}, \end{aligned}$$

where $\Lambda_t^{j,k} = \partial_t \Lambda^{j,k}(t,\xi;\gamma)$, $W_0^{j,k} = W_0^{j,k}(t,\xi;\gamma)$ and $W_{0t}^{j,k} = \partial_t W_0^{j,k}(t,\xi;\gamma)$. Put

$$\hat{f}_\varepsilon(t,\xi) = P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)$$

$$P^{j,k}(t,\tau,\xi;\varepsilon) = p^{j,k}(t,\tau,\xi) + q^{j,k}(t,\tau,\xi) + r^{j,k}(t,\tau,\xi;\varepsilon),$$

where $q^{j,k}(t,\tau,\xi) \in \mathscr{S}_{1,0}^{1}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^0))$ is positively homogeneous of degree 1 for $|\xi| \geq 1$ and $r^{j,k}(t,\tau,\xi;\varepsilon) \in \mathscr{S}_{1,0}^{1,-1}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathscr{N}^0))$ uniformly in ε. Then we have

$$(3.22) \quad \partial_t \mathscr{E}^{j,k}(t,\xi;v;\gamma,A)$$

$$\begin{aligned}
&\leq -\sum_{l=1}^{2} [A\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} |p_l^{j,k} v|^2 \\
&\qquad - 2\,\mathrm{Im}\{e^{-A\Lambda^{j,k}}(D_t - \lambda_l^{j,k}) p_l^{j,k} v \cdot \overline{(p_l^{j,k} v)}\}] \\
&\quad - \{A\Lambda_t^{j,k}(W_0^{j,k})^2 - 2(W_0^{j,k})^3\} e^{-A\Lambda^{j,k}} |v|^2 \\
&\quad - \mathrm{Im}\left\{(W_0^{j,k})^2 e^{-A\Lambda^{j,k}} \sum_{l}^{2} p_l^{j,k} v \cdot \bar{v}\right\} \\
&\leq -\sum_{l=1}^{2} e^{-A\Lambda^{j,k}} \Bigg[A\Lambda_t^{j,k} |p_l^{j,k} v|^2 - (\Lambda_t^{j,k})^{-1} |\hat{f}_\varepsilon|^2 - 3\Lambda_t^{j,k} |p_l^{j,k} v|^2 \\
&\qquad - 3(\Lambda_t^{j,k})^{-1} \left| \left(sub\ \sigma(P^{j,k})(t,D_t,\xi) + (-1)^l \frac{i}{2} \partial_t(\lambda_2^{j,k} - \lambda_1^{j,k}) \right) v \right|^2 \\
&\qquad - 3(\Lambda_t^{j,k})^{-1} |r^{j,k} v|^2 \Bigg] - \{A\Lambda_t^{j,k}(W_0^{j,k})^2 - 2(W_0^{j,k})^3\} e^{-A\Lambda^{j,k}} |v|^2 \\
&\quad + (\Lambda_t^{j,k})^{-1}(W_0^{j,k})^2 e^{-A\Lambda^{j,k}} \sum_{l=1}^{2} |p_l^{j,k} v|^2 + \frac{1}{2}\Lambda_t^{j,k}(W_0^{j,k})^2 e^{-A\Lambda^{j,k}} |v|^2 \\
&\leq 2(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} |\hat{f}_\varepsilon|^2 - \sum_{l=1}^{2} (A-4)\Lambda_t^{j,k} e^{-A\Lambda^{j,k}} |p_l^{j,k} v|^2 \\
&\quad + 12(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} |sub\ \sigma(P^{j,k}) v|^2 \\
&\quad + 3(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} |(\lambda_{1t}^{j,k} - \lambda_{2t}^{j,k}) v|^2 + 6(\Lambda_t^{j,k})^{-1} e^{-A\Lambda^{j,k}} |r^{j,k} v|^2 \\
&\quad - (A - 5/2)\Lambda_t^{j,k}(W_0^{j,k})^2 e^{-A\Lambda^{j,k}} |v|^2
\end{aligned}$$

since

$$(\tau - \lambda_l^{j,k}(t,\xi)) \circ p_l^{j,k}(t,\tau,\xi)$$
$$= P^{j,k}(t,\tau,\xi;\varepsilon) - q^{j,k}(t,\tau,\xi) - r^{j,k}(t,\tau,\xi;\varepsilon) - i\partial_t p_l^{j,k}(t,\tau,\xi),$$
$$-i\partial_t p_l^{j,k}(t,\tau,\xi) = (-1)^l \frac{i}{2}\partial_t(\lambda_1^{j,k}(t,\xi) - \lambda_2^{j,k}(t,\xi)) - \frac{i}{2}\partial_t\partial_\tau p^{j,k}(t,\tau,\xi) \quad (l=1,2),$$

where $p_l^{j,k} = p_l^{j,k}(t,D_t,\xi)$, $\lambda_l^{j,k} = \lambda_l^{j,k}(t,\xi)$, $sub\,\sigma(P^{j,k}) = sub\,\sigma(P^{j,k})(t,D_t,\xi)$, $\lambda_{lt}^{j,k} = \partial_t\lambda_l^{j,k}(t,\xi)$ and so forth. It is easy to see that

$$|(\lambda_{1t}^{j,k}(t,\xi) - \lambda_{2t}^{j,k}(t,\xi))v(t,\xi)|^2 \le 4(\Lambda_t^{j,k})^2 \sum_{l=1}^{2} |p_l^{j,k}v|^2 + 2(\Lambda_t^{j,k})^2|v|^2, \tag{3.23}$$

$$|r^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \le C\left\{|\xi|^{-2}\sum_{l=1}^{2}|p_l^{j,k}v|^2 + |v|^2\right\} \tag{3.24}$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^0)$ with $|\xi| \ge 1$ and $\varepsilon \in (0,1]$,

where $C > 0$. First assume that

$$\min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,\delta_1+1]} |t-s|, 1\right\} \le \langle\xi\rangle_\gamma^{-1/2}. \tag{3.25}$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \ge \langle\xi\rangle_\gamma^{1/2}/\sqrt{2}.$$

Therefore, there is $A_0 > 0$ satisfying

$$\partial_t \mathscr{E}^{j,k}(t,\xi;v;\gamma,A) \le 2|\hat{f}_\varepsilon(t,\xi)|^2 \tag{3.26}$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^0)$ with $|\xi| \ge 1$, $\varepsilon \in (0,1]$ and $A \ge A_0$ if (3.25) is satisfied. Next assume that

$$\min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,\delta_1+1]} |t-s|, 1\right\} \ge \langle\xi\rangle_\gamma^{-1/2}.$$

Then we have

$$W_0^{j,k}(t,\xi;\gamma) \ge \left(\sqrt{2}\min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,\delta_1+1]} |t-s|, 1\right\}\right)^{-1}.$$

Lemma 3.1 (i) and (3.22)–(3.24) prove that (3.26) is valid for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^0)$ with $|\xi| \ge 1$, $\varepsilon \in (0,1]$ and $A \ge A_0$, with a modification of A_0 if necessary. Repeating the same argument as in Lemma 3.3, we have the following

Lemma 3.4. *Assume that* $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 2$. *Then for* $\mu \in \mathbf{N}$ *and* $\kappa \in \mathbf{R}$ *there are* $\nu_{j,k} > 0$ *and* $C_\mu > 0$ *such that*

$$\sum_{l=0}^{\mu} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_t^l v(t,\xi)|^2$$

$$\le C_\mu \Biggl\{ \sum_{l=0}^{1} \langle\xi\rangle_\gamma^{2\mu+2\kappa+2+\nu_{j,k}-2l} |(D_t^l v)(0,\xi)|^2$$

$$+ \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa+\nu_{j,k}} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$

$$+ \sum_{l=0}^{\mu-2} \langle\xi\rangle_\gamma^{2\mu+2\kappa-4-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \Biggr\}$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \mathcal{N}^0)$ *with* $|\xi| \ge \gamma \ge 1$, $\varepsilon \in (0,1]$ *and* $v \in C^\infty([0,\delta_1]; L^\infty(\mathbf{R}^n))$, *where* $\sum_{l=0}^{\mu-2} \cdots = 0$ *when* $\mu = 1$ *and the* $\nu_{j,k}$ *do not depend on* μ.

(III) Now consider the case where $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 1$. Define

$$\mathscr{E}^{j,k}(t,\xi;v;A) = e^{-At}|v(t,\xi)|^2$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j$ with $|\xi| \ge 1$, $A \ge 1$ and $v(t,\xi) \in C([0,\delta_1]; L^\infty(\mathbf{R}^n))$. Then we have

$$D_t \mathscr{E}^{j,k}(t,\xi;v;A) = iAe^{-At}|v(t,\xi)|^2 + 2i\,\mathrm{Im}\{e^{-At}p^{j,k}v \cdot \bar{v}\},$$

where $p^{j,k} = p^{j,k}(t,D_t,\xi)$ $(= D_t - \lambda^{j,k}(t,\xi))$. Applying the same argument as in the proof of Lemma 3.3, we can prove the following

Lemma 3.5. *Assume that* $1 \le j \le N_0$, $1 \le k \le r(j)$ *and* $m(j,k) = 1$. *Then for* $\mu \in \mathbf{Z}_+$ *and* $\kappa \in \mathbf{R}$ *there is* $C_\mu > 0$ *such that*

$$\sum_{l=0}^{\mu} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_t^l v(t,\xi)|^2$$

$$\le C_\mu \Biggl\{ \langle\xi\rangle_\gamma^{2\mu+2\kappa} |v(0,\xi)|^2 + \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa} |P^{j,k}(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$

$$+ \sum_{l=0}^{\mu-1} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2-2l} |D_t^l P^{j,k}(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \Biggr\}$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}_j$ *with* $|\xi| \geq \gamma \geq 1$, $\varepsilon \in (0,1]$ *and* $v \in C^\infty([0,\delta_1]; L^\infty(\mathbf{R}^n))$, *where* $\sum_{l=0}^{\mu-1} \cdots = 0$ *when* $\mu = 0$.

Let $f(t,x) \in C^\infty(\mathbf{R}; \mathscr{S}(\mathbf{R}^n_x))$ satisfy $\operatorname{supp} f \subset \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\ t \geq 0\}$, and consider the Cauchy problem

$$\text{(CP)}' \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x), \\ u(t,x)|_{t<0} = 0. \end{cases}$$

Put

$$\tilde{\Gamma}_1 = \Gamma_1, \quad \tilde{\Gamma}_j = \Gamma_j \Big\backslash \bigcup_{l=1}^{j-1} \Gamma_l \quad (2 \leq j \leq N_0),$$

$$\tilde{\mathscr{N}} = \bigcup_{j=1}^{N_0} \left(\bigcup_{1 \leq k \leq r(j),\, m(j,k)=3} \mathscr{N}^{j,k} \right) \cup \mathscr{N}_2(p) \cup \{0\}.$$

Let $v_0(t,\xi;\varepsilon)$ $(\equiv v(t,\xi)) \in C^\infty(\mathbf{R}; \mathscr{S}(\mathbf{R}^n_\xi))$ satisfy $v_0(t,\xi;\varepsilon)|_{t<0} = 0$, and define

$$v_{k+1}(t,\xi;\varepsilon) = P^{j,r(j)-k}(t,D_t,\xi;\varepsilon)v_k(t,\xi;\varepsilon)$$
$$\text{for } 1 \leq j \leq N_0,\ \xi \in \tilde{\Gamma}_j \backslash \tilde{\mathscr{N}} \text{ and } 0 \leq k \leq r(j)-1.$$

Then it follows from Lemmas 3.3–3.5 that for $1 \leq j \leq N_0$, $0 \leq k \leq r(j)-1$, $\mu \geq m(j,r(j)-k)$, $\kappa \in \mathbf{R}$ and $(t,\xi) \in [0,\delta_1] \times (\tilde{\Gamma}_j \backslash \tilde{\mathscr{N}})$ with $|\xi| \geq \gamma \geq 1$

$$\sum_{l=0}^{\mu} \int_0^t \langle \xi \rangle_\gamma^{2\mu+2\kappa-2l} |D_s^l v_k(s,\xi;\varepsilon)|^2\, ds$$
$$\leq C_\mu \sum_{l=0}^{\mu-m(j,r(j)-k)} \int_0^t \langle \xi \rangle_\gamma^{2\mu+2\kappa+\tilde{v}_{j,k}-2l} |D_s^l v_{k+1}(s,\xi;\varepsilon)|^2\, ds,$$

where $C_\mu > 0$, $\tilde{v}_{j,k} = 0$ if $m(j,r(j)-k) = 1$ and $\tilde{v}_{j,k} = v_{j,r(j)-k}$ if $m(j,r(j)-k) = 2$ or 3, since

$$\int_0^t \left(\int_0^{s_1} |g(s,\xi)|\, ds \right) ds_1 = \int_0^t \left(\int_s^t |g(s,\xi)|\, ds_1 \right) ds$$
$$= \int_0^t (t-s)|g(s,\xi)|\, ds \leq \delta_1 \int_0^t |g(s,\xi)|\, ds$$

for $t \in [0,\delta_1]$. This yields

$$\sum_{l=0}^{\mu}\int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_s^l v_0(s,\xi;\varepsilon)|^2\,ds$$
$$\le C_\mu\sum_{l=0}^{\mu-m}\int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l+\tilde{\nu}}|D_s^l v_{r(j)}(s,\xi;\varepsilon)|^2\,ds$$

for $1\le j\le N_0$, $\mu\ge m$, $\kappa\in\mathbf{R}$ and $(t,\xi)\in[0,\delta_1]\times(\tilde{\Gamma}_j\backslash\tilde{\mathscr{N}})$ with $|\xi|\ge\gamma\ge 1$, where $C_\mu>0$ and $\tilde{\nu}=\max_{1\le j\le N_0}(\tilde{\nu}_{j,0}+\tilde{\nu}_{j,1}+\cdots+\tilde{\nu}_{j,r(j)-1})$. By (2.4) we can see that there are $C>0$ and $C_N>0$ $(N=0,1,2,\dots)$ satisfying

$$\sum_{l=0}^{m}\int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l}|D_s^l v(s,\xi)|^2\,ds$$
$$\le C\int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa+\tilde{\nu}}|P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$
$$+C_N\sum_{l=0}^{m-1}\int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l-N}|D_s^l v(s,\xi)|^2\,ds$$

for $\kappa\in\mathbf{R}$, $(t,\xi)\in[0,\delta_1]\times(\mathbf{R}^n\backslash\tilde{\mathscr{N}})$ with $|\xi|\ge\gamma\ge 1$ and $N=0,1,2,\dots$. Therefore, taking $\gamma_0=2C_1$ and modifying $\tilde{\nu}$ if necessary, we have

$$\sum_{l=0}^{m}\int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa-2l}|D_s^l v(s,\xi)|^2\,ds \tag{3.27}$$
$$\le 2C\int_0^t \langle\xi\rangle_\gamma^{2m+2\kappa+\tilde{\nu}}|P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$

for $v(t,\xi)\in C^\infty(\mathbf{R};\mathscr{S}(\mathbf{R}^n_\xi))$ with $v(t,\xi)|_{t<0}=0$ if $\kappa\in\mathbf{R}$, $(t,\xi)\in[0,\delta_1]\times(\mathbf{R}^n\backslash\tilde{\mathscr{N}})$, $\varepsilon\in(0,1]$ and $|\xi|\ge\gamma\ge\gamma_0$.

Lemma 3.6. *There are $C_\mu>0$ $(\mu\ge m)$ such that*

$$\sum_{l=0}^{\mu}\int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l}|D_s^l v(s,\xi)|^2\,ds \tag{3.28}$$
$$\le C_\mu\sum_{l=0}^{\mu-m}\int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l+\tilde{\nu}}|D_s^l P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds$$

for $\mu\ge m$, $\kappa\in\mathbf{R}$, $v(t,\xi)\in C^\infty(\mathbf{R};\mathscr{S}(\mathbf{R}^n_\xi))$ with $v(t,\xi)|_{t<0}=0$, $(t,\xi)\in[0,\delta_1]\times(\mathbf{R}^n\backslash\tilde{\mathscr{N}})$ with $|\xi|\ge\gamma\ge\gamma_0$ and $\varepsilon\in(0,1]$.

Proof. Let $M \geq m$, and assume that (3.28) is valid for $\mu = M$. We have

$$D_t^m v(t,\xi) = -\sum_{l=0}^{m-1} a_{m-l}(t,\xi;\varepsilon) D_t^l v(t,\xi) + P(t,D_t,\xi;\varepsilon)v(t,\xi),$$

where $a_l(t,\xi;\varepsilon) = \sum_{|\alpha|\leq l} a_{l,\alpha}(t;\varepsilon)\xi^\alpha \in S_{1,0}^l([0,\delta_1]\times\mathbf{R}^n)$ uniformly in $\varepsilon \in (0,1]$ $(1 \leq l \leq m)$. By induction we can easily show that for $h \in \mathbf{Z}_+$ there are symbols $a_{m+h-l}^h(t,\xi;\varepsilon) \in S_{1,0}^{m+h-l}([0,\delta_1]\times\mathbf{R}^n)$ and $b_{h-\nu}^h(t,\xi;\varepsilon) \in S_{1,0}^{h-\nu}([0,\delta_1]\times\mathbf{R}^n)$ $(0 \leq l \leq m-1,\ 0 \leq \nu \leq h)$ uniformly in $\varepsilon \in (0,1]$ satisfying

$$D_t^{m+h} v(t,\xi) = \sum_{l=0}^{m-1} a_{m+h-l}^h(t,\xi;\varepsilon) D_t^l v(t,\xi) + \sum_{l=0}^{h} b_{h-l}^h(t,\xi;\varepsilon) D_t^l P(t,D_t,\xi;\varepsilon)v(t,\xi).$$

This, with (3.27) and (3.28) for $\mu = M$, proves (3.28) for $\mu = M+1$. □

(IV) Let us derive energy estimates for $|\xi| \leq \gamma$. Define

$$\mathscr{E}^0(t,\xi;v;\gamma,A) = \sum_{l=0}^{m-1} e^{-At}\langle\xi\rangle_\gamma^{2m-2-2l}|D_t^l v(t,\xi)|^2$$

for $(t,\xi) \in [0,\delta_1]\times\mathbf{R}^n$ with $|\xi| \leq \gamma$ and $v(t,\xi) \in C^m([0,\delta_1];L^\infty(\mathbf{R}^n))$, where $A \geq 1$ and $\gamma \geq \gamma_0$. Then we have

$$D_t\mathscr{E}^0(t,\xi;v;\gamma,A) = \sum_{l=0}^{m-1} iAe^{-At}\langle\xi\rangle_\gamma^{2m-2-2l}|D_t^l v(t,\xi)|^2 + 2ie^{-At}\,\mathrm{Im}\{D_t^m v \cdot (\overline{D_t^{m-1}v})\}$$

$$+ \sum_{l=0}^{m-2} 2i\langle\xi\rangle_\gamma^{2m-2-2l} e^{-At}\,\mathrm{Im}\{D_t^{l+1}v \cdot (\overline{D_t^l v})\}.$$

Since $P(t,\tau,\xi;\varepsilon) - \tau^m \in \mathscr{S}_{1,0}^{m-1,1}(\mathbf{R}\times\mathbf{R}^n)$ uniformly in ε, there is $C_0 > 0$ such that

$$\partial_t\mathscr{E}^0(t,\xi;v;\gamma,A) \leq 4A^{-1}e^{-At}|P(t,D_t,\xi;\varepsilon)v(t,\xi)|^2$$

if $A \geq C_0\gamma$ and $|\xi| \leq \gamma$. This yields

$$\sum_{l=0}^{m-1} \langle\xi\rangle_\gamma^{2m+2\kappa-2-2l}|D_t^l v(t,\xi)|^2$$

$$\leq C_\gamma\left\{\sum_{l=0}^{m-1}\langle\xi\rangle_\gamma^{2m+2\kappa-2-2l}|(D_t^l v)(0,\xi)|^2 + \int_0^t \langle\xi\rangle_\gamma^{2\kappa}|P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\,ds\right\}$$

for $(t,\xi) \in [0,\delta_1] \times \mathbf{R}^n$ with $|\xi| \le \gamma$, $\varepsilon \in (0,1]$ and $v(t,\xi) \in C^m([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where C_γ is a positive constant depending on γ. Similarly, for $\mu \ge m-1$ there are $C_{\gamma,\mu} > 0$ $(\mu \ge m-1)$ such that

$$\sum_{l=0}^{\mu} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2l} |D_t^l v(t,\xi)|^2 \le C_{\gamma,\mu}\Bigg\{ \sum_{l=0}^{m-1} \langle\xi\rangle_\gamma^{2m+2\kappa-2l} |(D_t^l v)(0,\xi)|^2 + \int_0^t \langle\xi\rangle_\gamma^{2\mu+2\kappa-2m+2} |P(s,D_s,\xi;\varepsilon)v(s,\xi)|^2\, ds + \sum_{l=0}^{\mu-m} \langle\xi\rangle_\gamma^{2\mu+2\kappa-2m-2l} |D_t^l P(t,D_t,\xi;\varepsilon)v(t,\xi)|^2 \Bigg\}$$

for $(t,\xi) \in [0,\delta_1] \times \mathbf{R}^n$ with $|\xi| \le \gamma$, $\varepsilon \in (0,1]$ and $v(t,\xi) \in C^m([0,\delta_1]; L^\infty(\mathbf{R}^n))$, where $\sum_{l=0}^{-1} \cdots = 0$. This, together with Lemmas 3.3–3.6, yields the following

LEMMA 3.7. *There are $\gamma_0 \ge 1$, $C_{\gamma,\mu} > 0$ $(\gamma \ge \gamma_0, \mu \ge m)$ and $\nu_0 > 0$ such that*

$$\sum_{l=0}^{\mu} \|\langle D_x\rangle_\gamma^{\mu+\kappa-l} D_t^l u(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)} \le C_{\gamma,\mu} \sum_{l=0}^{\mu-m} \|\langle D_x\rangle_\gamma^{\mu+\kappa-m-l+\nu_0} D_t^l P(t,D_t,D_x;\varepsilon)u(t,x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)} \tag{3.29}$$

if $\mu \ge m$, $\gamma \ge \gamma_0$, $\varepsilon \in (0,1]$ and $u(t,x) \in C^\infty(\mathbf{R}; H^\infty(\mathbf{R}^n))$ with $u(t,x)|_{t<0} = 0$. Here $H^s(\mathbf{R}^n)$ denotes the Sobolev space of order s and $H^\infty(\mathbf{R}^n) = \bigcap_{s\in\mathbf{R}} H^s(\mathbf{R}^n)$ and

$$\|f(t,x)\|_{L^2([0,\delta_1]\times\mathbf{R}^n)} = \left(\int_{[0,\delta_1]\times\mathbf{R}^n} |f(t,x)|^2\, dtdx \right)^{1/2}.$$

REMARK. (3.29) is valid, replacing $P(t,D_t,D_x;\varepsilon)$ by $P(t,D_t,D_x)$.

Let $f(t,x) \in C^\infty([0,\infty); H^\infty(\mathbf{R}^n))$ satisfy $(D_t^j f)(0,x) = 0$ for $j \in \mathbf{Z}_+$. Then, it follows from the unique existence theorem for ordinary differential equations and the proof of Lemma 3.7 with $P(t,D_t,D_x;\varepsilon)$ replaced by $P(t,D_t,D_x)$ that the Cauchy problem

$$\begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\delta_1]\times\mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1) \end{cases} \tag{CP$_0$}$$

has a unique solution $u(t,x) \in C^\infty([0,\delta_1]; H^\infty(\mathbf{R}^n))$. We note that $(\mathrm{CP})_0$ has a unique solution $u(t,x) \in C^\infty([0,\delta_1]; H^\infty(\mathbf{R}^n))$ even if $P(t,D_t,D_x)$ is replaced by $P(t,D_t,D_x;\varepsilon)$.

Lemma 3.8. *Let* $u \in C^\infty((-\infty,\delta_1] \times \mathbf{R}^n)$ *satisfy* $u(t,x)|_{t<0} = 0$, *and let* $(t_0,x^0) \in [0,\delta_1] \times \mathbf{R}^n$. *Then* $(t_0,x^0) \notin \operatorname{supp} u$ *if*

$$K^-_{(t_0,x^0)} \cap \operatorname{supp} P(t,D_t,D_x)u = \emptyset. \tag{3.30}$$

Proof. We extend $u(t,x)$ to a function in $C^\infty(\mathbf{R}^{n+1})$. Choose $R > 0$ so that

$$K^-_{(t_0,x^0)} \subset \{(t,x) \in [0,\delta_1] \times \mathbf{R}^n;\ |x| \le R\}.$$

Assume that (3.30) is valid. Let $\Theta(t)$ be a function in $\mathscr{E}^{\{\kappa_0\}}(\mathbf{R})$ satisfying

$$\Theta(t) = \begin{cases} 1 & \text{if } t \le 3/2, \\ 0 & \text{if } t \ge 2. \end{cases}$$

Put

$$F_R(t,x) = \Theta(|x|-R)P(t,D_t,D_x)u(t,x) + [P, \Theta(|x|-R)]u(t,x),$$

where $[A,B] = AB - BA$. Then we have

$$P(t,D_t,D_x)(\Theta(|x|-R)u(t,x)) = F_R(t,x).$$

Note that $F_R(t,x)|_{t<0} = 0$. It is easy to see that there is a unique solution $v_R(t,x) \in C^\infty((-\infty,\delta_1]; H^\infty(\mathbf{R}^n))$ satisfying

$$(\mathrm{CP})_R \qquad \begin{cases} P(t,D_t,D_x)v_R(t,x) = F_R(t,x) & \text{in } (-\infty,\delta_1] \times \mathbf{R}^n, \\ v_R(t,x)|_{t<0} = 0. \end{cases}$$

Therefore, we have $v_R(t,x) = \Theta(|x|-R)u(t,x)$ for $t \in (-\infty,\delta_1]$. Choose $\rho^1(t) \in \mathscr{E}^{\{\kappa_0\}}(\mathbf{R})$ and $\rho^n(x) \in \mathscr{E}^{\{\kappa_0\}}(\mathbf{R}^n)$ so that $\rho^1(t) \ge 0$, $\int_{-\infty}^\infty \rho^1(t)\,dt = 1$, $\operatorname{supp} \rho^1 \subset \{t \in \mathbf{R};\ 0 \le t \le 1\}$, $\rho^n(x) \ge 0$, $\int_{\mathbf{R}^n} \rho^n(x)\,dx = 1$, $\operatorname{supp} \rho^n \subset \{x \in \mathbf{R}^n;\ |x| \le 1\}$. Here we say that $f(x) \in \mathscr{E}^{\{\kappa\}}(\mathbf{R}^n)$ if for any $T > 0$ there are $h > 0$ and $C_T > 0$ satisfying

$$|\partial_x^\alpha f(x)| \le C_T h^{|\alpha|}(|\alpha|!)^\kappa \quad \text{for } \alpha \in (\mathbf{Z}_+)^n \text{ and } x \in \mathbf{R}^n \text{ with } |x| \le T.$$

For $\varepsilon > 0$ we define

$$F_{R,\varepsilon}(t,x) = \int_{\mathbf{R}^{n+1}} \rho^1_\varepsilon(t-s)\rho^n_\varepsilon(x-y)F_R(s,y)\,dsdy,$$

for $(t, x) \in \mathbf{R}^{n+1}$, where $\rho_\varepsilon^1(t) = \varepsilon^{-1}\rho^1(t/\varepsilon)$ and $\rho_\varepsilon^n(x) = \varepsilon^{-n}\rho^n(x/\varepsilon)$. Then we have $F_{R,\varepsilon}(t, x) \in \mathscr{E}^{\{\kappa_0\}}(\mathbf{R}^{n+1})$ and

$$\operatorname{supp} F_{R,\varepsilon}(t, x) \subset \{(t, x) \in \mathbf{R}^{n+1};\ t \geq 0 \text{ and } |x| \leq R + 2 + \varepsilon\}.$$

Moreover, we have

$$F_{R,\varepsilon}(t, x) \to F_R(t, x) \quad \text{in } C^\infty(\mathbf{R}; C_0^\infty(\mathbf{R}^n)) \text{ as } \varepsilon \downarrow 0.$$

It follows from [3] that the Cauchy problem

$$(\mathrm{CP})_{R,\varepsilon} \qquad \begin{cases} P(t, D_t, D_x; \varepsilon) v_{R,\varepsilon}(t, x) = F_{R,\varepsilon}(t, x) & \text{in } \mathbf{R}^{n+1}, \\ v_{R,\varepsilon}(t, x)|_{t<0} = 0 \end{cases}$$

has a unique solution $v_{R,\varepsilon}(t, x)$ in $\mathscr{E}^{\{\kappa_0\}}(\mathbf{R}^{n+1})$ and that $(t_0, x^0) \notin \operatorname{supp} v_{R,\varepsilon}$ if $\operatorname{supp} F_{R,\varepsilon} \cap K^-_{(t_0,x^0)} = \emptyset$. More precisely, we have

$$\operatorname{supp} v_{R,\varepsilon} \subset \{(t, x) \in \mathbf{R} \times \mathbf{R}^n;\ (t, x) \in K^+_{(s,y)} \text{ for some } (s, y) \in \operatorname{supp} F_{R,\varepsilon}\}.$$

For $\varepsilon, \varepsilon' \in (0, 1]$ with $\varepsilon' \leq \varepsilon$ we put $w_{R,\varepsilon,\varepsilon'}(t, x) = v_{R,\varepsilon}(t, x) - v_{R,\varepsilon'}(t, x)$. Then we have

$$P(t, D_t, D_x; \varepsilon) w_{R,\varepsilon,\varepsilon'}(t, x) = F_{R,\varepsilon}(t, x) - F_{R,\varepsilon'}(t, x) + \sum_{j=3}^{m} \sum_{|\alpha| \leq j-3} (a_{j,\alpha}(t; \varepsilon') - a_{j,\alpha}(t; \varepsilon)) D_t^{m-j} D_x^\alpha v_{R,\varepsilon'}(t, x).$$

Applying Lemma 3.7 we can see that there are $C_\mu > 0$ $(\mu \geq m)$ satisfying

$$(3.31) \quad \sum_{l=0}^{\mu} \|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-l} D_t^l w_{R,\varepsilon,\varepsilon'}(t, x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}$$
$$\leq C_\mu \Bigg\{ \sum_{l=0}^{\mu-m} \|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-m-l+\nu_0} D_t^l (F_{R,\varepsilon}(t, x) - F_{R,\varepsilon'}(t, x))\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)}$$
$$+ \sup_{\substack{t\in[0,\delta_1],\, 3\leq j\leq m \\ |\beta|\leq j-3,\, h\leq\mu-m}} |D_t^h (a_{j,\beta}(t; \varepsilon') - a_{j,\beta}(t; \varepsilon))|^2$$
$$\times \sum_{l=0}^{\mu-m} \|\langle D_x\rangle_{\gamma_0}^{\mu+\kappa-m-l-3+2\nu_0} D_t^l F_{R,\varepsilon'}(t, x)\|^2_{L^2([0,\delta_1]\times\mathbf{R}^n)} \Bigg\}$$

for $\mu \in \mathbf{N}$ with $\mu \geq m$ and $\kappa \in \mathbf{R}$. Indeed, we also apllied Lemma 3.7 to $v_{R,\varepsilon'}(t, x)$ in order to obtain (3.31). (3.31) yields

$$v_{R,\varepsilon}(t,x) \to v_R(t,x) \quad \text{in } C^\infty([0,\delta_1]; H^\infty(\mathbf{R}^n)) \text{ as } \varepsilon \downarrow 0,$$

$$\begin{aligned}&\operatorname{supp} v_R \cap (-\infty, \delta_1] \\ &\qquad \subset \{(t,x) \in [0,\delta_1] \times \mathbf{R}^n;\ (t,x) \in K^+_{(s,y)} \text{ for some } (s,y) \in \operatorname{supp} F_R\}.\end{aligned}$$

Since

$$\operatorname{supp}[P, \Theta(|x| - R)]u(t,x) \subset \left\{(t,x) \in [0,\infty) \times \mathbf{R}^n;\ R + \frac{3}{2} \le |x| \le R + 2\right\},$$

we have

$$K^-_{(t_0,x^0)} \cap \operatorname{supp} F_R = \emptyset,$$

which proves $(t_0, x^0) \notin \operatorname{supp} v_R$ and the lemma. □

For $f(t,x) \in C^\infty(\mathbf{R}^{n+1})$ with $f(t,x)|_{t<0} = 0$ we consider

$$(\mathrm{CP})_0' \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } (-\infty,\delta_1] \times \mathbf{R}^n, \\ u(t,x)|_{t<0} = 0. \end{cases}$$

Put $f_R(t,x) = \Theta(|x| - R)f(t,x)$ for $R > 0$, and let $u_R(t,x)$ be a solution to $(\mathrm{CP})_0'$ with $f(t,x)$ replaced by $f_R(t,x)$. Then we have

$$P(t,D_t,D_x)(u_{R'}(t,x) - u_R(t,x)) = (\Theta(|x| - R') - \Theta(|x| - R))f(t,x),$$

where $R' \ge R > 0$. Define

$$M_{\delta_1} = \sup_{\substack{t \in [0,\delta_1],\, 1 \le j \le m \\ \xi \in S^{n-1}}} |\lambda_j(t,\xi)|,$$

$$K_{\delta_1} = \{(t,x) \in \mathbf{R}^{n+1};\ t \ge |x|/M_{\delta_1}\}.$$

It is easy to see that

$$K^+_{(t_0,x^0)} \cap [0,\delta_1] \times \mathbf{R}^n \subset \{(t_0,x^0)\} + K_{\delta_1}.$$

Lemma 3.8 implies that

$$u_{R+\delta_1 M_{\delta_1}}(t,x) = u_{R'+\delta_1 M_{\delta_1}}(t,x) \quad \text{if } t \le \delta_1 \text{ and } |x| \le R \le R'.$$

Therefore, we can define $u(t,x)$ by

$$u(t,x) = u_{R+\delta_1 M_{\delta_1}}(t,x) \quad \text{for } t \le \delta_1 \text{ and } |x| \le R,$$

and $u(t,x)$ ($\in C^\infty((-\infty,\delta_1]\times\mathbf{R}^n)$) satisfies $(\mathrm{CP})_0'$. Repeating the same argument as at the end of §2.3 of [6], we can construct solutions to the Cauchy problem (CP) with $[0,\infty)\times\mathbf{R}^n$ replaced by $[0,\delta_1]\times\mathbf{R}^n$ when $f(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$ and $u_j(x)\in C^\infty(\mathbf{R}^n)$ $(0\le j\le m-1)$, and finally we can complete the proof of Theorem 1.2.

References

[1] Atiyah, M. F., Resolution of singularities and division of distributions, Comm. Pure Appl. Math. **23** (1970), 145–150.
[2] Kumano-go, H., Pseudo-Differential Operators, MIT Press, Cambridge, 1982.
[3] Wakabayashi, S., Singularities of solutions of the Cauchy problem for hyperbolic systems in Gevrey classes, Japanese J. Math. **11** (1985), 157–201.
[4] Wakabayashi, S., Remarks on hyperbolic polynomials, Tsukuba J. Math. **10** (1986), 17–28.
[5] Wakabayashi, S., On the Cauchy problem for a class of hyperbolic operators whose coefficients depend only on the time variable, Tsukuba J. Math. **39** (2015), 121–163.
[6] Wakabayashi, S., On the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time dependent coefficients, Funkcialaj Ekvacioj **63** (2020), 345–418.
[7] Wakabayashi, S., On the Cauchy problem for hyperbolic operators with triple charecteristics whose coefficients depend only on the time variable II, Tsukuba J. Math. **49** (2025), 51–105.

Seiichiro Wakabayashi
(professor emeritus)
University of Tsukuba
E-mail: wkbysh@wb3.so-net.ne.jp