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ON THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS WHOSE COEFFICIENTS DEPEND ONLY ON THE TIME VARIABLE II

By

Seiichiro WAKABAYASHI

Abstract. In [11] we considered the Cauchy problem for hyperbolic operators with triple characteristics whose coefficients depend only on the time variable. And we gave sufficient conditions for C^∞ well-posedness. In this paper we shall show that the sufficient conditions given in [11] are also necessary under additional assumptions.

1. Introduction

In [11] we investigated the Cauchy problem for hyperbolic operators with triple characteristics whose coefficients depend only on the time variable. And we gave sufficient conditions for C^∞ well-posedness. This paper is the sequel to [11], and we shall prove that the sufficient conditions given in [11] are also necessary if the space dimension is less than 3 or if the coefficients are semi-algebraic functions of the time variable. Here we say that $h(t)$ is semi-algebraic if $h(t)$ is defined in a semi-algebraic set U in $\mathbf{R}$ and its graph $\{(t, h(t)) \in \mathbf{R}^2;\, t \in U\}$ is a semi-algebraic set (see, *e.g.*, [13]). For basic properties of semi-algebraic functions we refer to [13] and [14].

Let $m \in \mathbf{N}$ satisfy $m \geq 2$, and let $P(t, \tau, \xi) \equiv \tau^m + \sum_{j=1}^m \sum_{|\alpha| \leq j} a_{j,\alpha}(t) \tau^{m-j} \times \xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1, \ldots, \xi_n)$ of degree m whose coefficients $a_{j,\alpha}(t)$ belong to $C^\infty([0, \infty])$. We consider the Cauchy problem

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$$\text{(CP)} \qquad \begin{cases} P(t, D_t, D_x)u(t,x) = f(t,x) & \text{in } [0,\infty) \times \mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1) \end{cases}$$

in the framework of the space of C^∞ functions. For notations and terminologies in this paper we refer to [11]. Put

$$p(t,\tau,\xi) = \tau^m + \sum_{j=1}^m \sum_{|\alpha|=j} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha \ (\equiv P_m(t,\tau,\xi)),$$

$$P_k(t,\tau,\xi) = \sum_{j=m-k}^m \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha \quad (0 \le k \le m-1).$$

We assume throughout this paper that the following conditions (A), (H)$'$ and (T) are satisfied:

(A) $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j, j-1, j-2)$ are real analytic in $[0,\infty)$.

(H)$'$ $p(t,\tau,\xi)$ is hyerbolic with respect to $\vartheta = (1,0,\dots,0) \in \mathbf{R}^{n+1}$ for $t \in [0,\infty)$, *i.e.*,

$$p(t,\tau - i,\xi) \ne 0 \quad \text{for any } (t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times \mathbf{R}^n.$$

(T) The characteristic roots are at most triple, *i.e.*,

$$\partial_\tau^3 p(t,\tau,\xi) \ne 0 \quad \text{if } (t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times S^{n-1} \text{ and}$$
$$p(t,\tau,\xi) = \partial_\tau p(t,\tau,\xi) = \partial_\tau^2 p(t,\tau,\xi) = 0,$$

where $S^{n-1} = \{\xi \in \mathbf{R}^n;\ |\xi| = 1\}$. From (A) there are a complex neighborhood Ω of $[0,\infty)$ (in $\mathbf{C}$) and $\delta_0 > 0$ such that $[-\delta_0,\infty) \subset \Omega$, $\Omega \cap \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \le T\}$ is compact for any $T > 0$, and $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j)$ are regarded as analytic functions defined in Ω. Write

$$p(t,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,\xi)).$$

Put

$$\mu_{j,k}(t,\xi) = (\lambda_j(t,\xi) - \lambda_k(t,\xi))^2, \quad M = \binom{m}{2},$$

and define $\{D_l(t,\xi)\}_{1 \le l \le M}$ by

$$\tau^M + \sum_{l=1}^M D_l(t,\xi)\tau^{M-l} = \prod_{1 \le j < k \le m} (\tau + \mu_{j,k}(t,\xi)).$$

We note that $D_M(t,\xi)$ $(\equiv D(t,\xi))$ is the discriminant of $p(t,\tau,\xi)=0$ in τ. Putting $D_0(t,\xi)\equiv 1$, for each $\xi\in S^{n-1}$ there is $r(\xi)\in \mathbf{Z}_+$ such that $0\le r(\xi)\le M$ and

$$D_M(t,\xi)\equiv\cdots\equiv D_{M-r(\xi)+1}(t,\xi)\equiv 0 \quad \text{in } t,$$
$$D_{M-r(\xi)}(t,\xi)\not\equiv 0 \quad \text{in } t.$$

It is easy to see that

$$D_{M-r(\xi)}(t,\xi)=\prod_{\substack{1\le j<k\le m\\ \mu_{j,k}(t,\xi)\not\equiv 0 \text{ in } t}}\mu_{j,k}(t,\xi),$$

$$r(\xi)=\#\{(j,k);\ 1\le j<k\le m \text{ and } \mu_{j,k}(t,\xi)\equiv 0 \text{ in } t\}.$$

We define

$$\mathscr{R}_0(\xi)=\{(\mathrm{Re}\,\lambda)_+;\ \lambda\in\Omega \text{ and } D_{M-r(\xi)}(\lambda,\xi)=0\}\quad \text{for } \xi\in S^{n-1},$$

where $a_+=\max\{0,a\}$ for $a\in\mathbf{R}$. By Lemma 2.1 in [11] we may assume that for any $T>0$ there is $N_T\in\mathbf{Z}_+$ satisfying

$$\#(\mathscr{R}_0(\xi)\cap[0,T])\le N_T \quad \text{for } \xi\in S^{n-1},$$

modifying Ω if necessary. To describe conditions on the lower order terms we define the polynomials $h_j(t,\tau,\xi)$ $(\equiv h_j(t,\tau,\xi;p))$ of (τ,ξ) by

$$|p(t,\tau-i\gamma,\xi)|^2=\sum_{j=0}^m \gamma^{2j}h_{m-j}(t,\tau,\xi)\quad \text{for } (t,\tau,\xi)\in[0,\infty)\times\mathbf{R}\times\mathbf{R}^n \text{ and } \gamma\in\mathbf{R}.$$

Since $|p(t,\tau-i\gamma,\xi)|^2=\prod_{j=1}^m((\tau-\lambda_j(t,\xi))^2+\gamma^2)$, we have

$$h_k(t,\tau,\xi)=\sum_{1\le j_1<j_2<\cdots<j_k\le m}\prod_{l=1}^k(\tau-\lambda_{j_l}(t,\xi))^2 \quad (1\le k\le m). \tag{1.1}$$

The subprincipal symbol of $P(t,D_t,D_x)$ is defined by

$$sub\ \sigma(P)(t,\tau,\xi)=P_{m-1}(t,\tau,\xi)+\frac{i}{2}\partial_t\partial_\tau p(t,\tau,\xi).$$

To describe the Levi condition on the $(m-2)$-th order terms of P we have to define some quantities. Let $z^0\equiv(t_0,\tau_0,\xi^0)\in[0,\infty)\times\mathbf{R}\times S^{n-1}$ satisfy $(\partial_\tau^k p)(z^0)=0$ $(0\le k\le 2)$. Define a monic polynomial $p(t,\tau,\xi;z^0)$ of τ of degree 3 satisfying the following:

$$\begin{cases} p(t,\tau,\xi;z^0) \text{ is defined for } (t,\xi)\in\mathcal{U}(z^0) \text{ and } p(t,\tau,\xi) \text{ is divided} \\ \text{by } p(t,\tau,\xi;z^0) \text{ as polynomials of } \tau, \text{ and, putting } \tilde{p}(t,\tau,\xi;z^0) = \\ p(t,\tau,\xi)/p(t,\tau,\xi;z^0), \\ \quad \tau\in I(z^0) \quad \text{if } (t,\xi)\in\mathcal{U}(z^0),\ |\xi|=1 \text{ and } p(t,\tau,\xi;z^0)=0, \\ \quad \tilde{p}(t,\tau,\xi;z^0)\neq 0 \quad \text{if } (t,\xi)\in\mathcal{U}(z^0),\ |\xi|=1 \text{ and } \tau\in I(z^0), \end{cases}$$

where $\mathcal{U}(z^0)$ is a neighborhood of (t_0,ξ^0) and $I(z^0)$ is a neighborhood of τ_0. Then we write

$$p(t,\tau,\xi;z^0) = \tau^3 + a_1(t,\xi;z^0)\tau^2 + a_2(t,\xi;z^0)\tau + a_3(t,\xi;z^0).$$

We define

$$\begin{aligned} (1.2) \qquad Q(t,\tau,\xi;z^0) = P_{m-2}(t,\tau,\xi) &+ \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{1}{4}\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0)\cdot\partial_t\tilde{p}(t,\tau,\xi;z^0) \\ &+ \frac{i}{12}\partial_\tau^2\, sub\ \sigma(P)(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0) \\ &+ \frac{1}{24}(\partial_t\partial_\tau^2 p(t,\tau,\xi;z^0))^2\cdot\partial_\tau\tilde{p}(t,\tau,\xi;z^0) \\ &\text{for } (t,\xi)\in\mathcal{U}(z^0) \text{ and } \tau\in\mathbf{R}. \end{aligned}$$

We note that

$$(1.3) \quad Q(t,\tau,\xi;z^0) = P_1(t,\tau,\xi) + \frac{1}{6}\partial_t^2\partial_\tau^2 p(t,\tau,\xi) + \frac{i}{12}\partial_\tau^2 P_2(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi)$$

when $m=3$. In [9] we defined the sub-sub-principal symbol $sub^2\,\sigma(P)(t,\tau,\xi)$ of P by the right-hand side of (1.3).

THEOREM 1.1. *Assume that $n\le 2$, and that the conditions* (A), (H)$'$ *and* (T) *are satisfied. If the Cauchy problem* (CP) *is C^∞ well-posed and has finite propagation property, then for any compact interval $I\subset(0,\infty)$ the following Levi conditions* (L-1)$_I$ *and* (L-2)$_I$ *are satisfied*:

(L-1)$_I$ *There is $C>0$ such that*

$$\min\left\{\min_{s\in\mathcal{R}_0(\xi)}|t-s|,1\right\}|sub\ \sigma(P)(t,\tau,\xi)| \le Ch_{m-1}(t,\tau,\xi)^{1/2} \quad \textit{for } (t,\tau,\xi)\in I\times\mathbf{R}\times S^{n-1}.$$

(L-2)$_I$ *For any* $z^0 = (t_0, \tau_0, \xi^0) \in I \times \mathbf{R} \times S^{n-1}$ *with* $(\partial_\tau^k p)(z^0) = 0$ $(0 \le k \le 2)$, *there are* $\hat{\delta} > 0$, *a neighborhood* U *of* ξ^0 *and* $C > 0$ *such that*

$$\min\left\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^2, 1\right\} |Q(t, -a_1(t, \xi; z^0)/3, \xi; z^0)|$$
$$\le C h_{m-2}(t, -a_1(t, \xi; z^0)/3, \xi)^{1/2}$$
$$\text{for } (t, \xi) \in (I \cap [t_0 - \hat{\delta}, t_0 + \hat{\delta}]) \times (S^{n-1} \cap U).$$

Theorem 1.2. *Assume that the conditions* (H)$'$, (T) *and the following condition* (A)$'$ *are satisfied*:

(A)$'$ $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j, j-1, j-2)$ *are semi-algebraic in* $[0, \infty)$.

Then the conditions (L-1)$_{[0,T]}$ *and* (L-2)$_{[0,T]}$ *for any* $T > 0$ *are satisfied if the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property.*

The remainder of this paper is organized as follows. In §2 we shall give preliminary lemmas. We shall prove Theorems 1.1 and 1.2, applying the arguments as in [4] (see, also, [12]). In §3 we shall construct asymptotic solutions for triple characteristic factors. For double characteristic factors we shall construct asymptotic solutions in §4. In §5 we shall prove Theorem 1.1. Theorem 1.2 will be proved in §6.

2. Preliminaries

From the assumption (T) there are $\delta_1 > 0$, $N_0 \in \mathbf{N}$, $m(j,k) \in \mathbf{N}$, open cones Γ_j in $\mathbf{R}^n \backslash \{0\}$, $r(j) \in \mathbf{N}$, compact intervals $J_{j,k}$ and $p^{j,k}(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(j,k)}([0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\}))$ $(1 \le j \le N_0,\ 1 \le k \le r(j))$ such that $m(j,k) \le 3$, the $p^{j,k}(t,\tau,\xi)$ are monic polynomials of τ and positively homogeneous of degree $m(j,k)$ in $(\tau,\xi) \in \mathbf{R} \times (\bar{\Gamma}_j \backslash \{0\})$ such that $\bigcup_{j=1}^{N_0} \Gamma_j \supset S^{n-1}$, $J_{j,k} \cap J_{j,l} = \emptyset$ for $1 \le j \le N_0$ and $1 \le k < l \le r(j)$,

$$p(t,\tau,\xi) = \prod_{k=1}^{r(j)} p^{j,k}(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\bar{\Gamma}_j \cap S^{n-1}), \tag{2.1}$$

$\tau \in J_{j,k}$ if $1 \le j \le N_0$, $1 \le k \le r(j)$, $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \cap S^{n-1})$, $\tau \in \mathbf{C}$ and $p^{j,k}(t,\tau,\xi) = 0$, and for each (j,k) with $1 \le j \le N_0$ and $1 \le k \le r(j)$ there is $(\hat{\tau}, \hat{\xi}) \in \mathbf{R} \times (\Gamma_j \cap S^{n-1})$ satisfying

$$(\partial_\tau^\mu p^{j,k})(0, \hat{\tau}, \hat{\xi}) = 0 \quad (0 \le \mu \le m(j,k) - 1).$$

We write

$$p(t,\tau,\xi) = \prod_{l=1}^{m}(\tau - \lambda_l(t,\xi)),$$

$$p^{j,k}(t,\tau,\xi) = \prod_{l=1}^{m(j,k)}(\tau - \lambda_l^{j,k}(t,\xi)).$$

Fix j so that $1 \le j \le N_0$. For $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \backslash \{0\})$ we write

$$p^{j,k}(t,\tau,\xi) = \tau^3 + a_1^{j,k}(t,\xi)\tau^2 + a_2^{j,k}(t,\xi)\tau + a_3^{j,k}(t,\xi),$$

$$\hat{p}^{j,k}(t,\tau,\xi) = p^{j,k}(t,\tau - a_1^{j,k}(t,\xi)/3,\xi) = \tau^3 - \hat{a}_2^{j,k}(t,\xi)\tau + \hat{a}_3^{j,k}(t,\xi)$$
$$\text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 3,$$

$$p^{j,k}(t,\tau,\xi) = \tau^2 + a_1^{j,k}(t,\xi)\tau + a_2^{j,k}(t,\xi),$$

$$\hat{p}^{j,k}(t,\tau,\xi) = p^{j,k}(t,\tau - a_1^{j,k}(t,\xi)/2,\xi) = \tau^2 - \hat{a}_2^{j,k}(t,\xi)$$
$$\text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 2.$$

Then we have

$$\hat{a}_2^{j,k}(t,\xi) = a_1^{j,k}(t,\xi)^2/3 - a_2^{j,k}(t,\xi) \ (\ge 0),$$

$$\hat{a}_3^{j,k}(t,\xi) = 2a_1^{j,k}(t,\xi)^3/27 - a_1^{j,k}(t,\xi)a_2^{j,k}(t,\xi)/3 + a_3^{j,k}(t,\xi)$$
$$\text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 3,$$

$$\hat{a}_2^{j,k}(t,\xi) = a_1^{j,k}(t,\xi)^2/4 - a_2^{j,k}(t,\xi)$$
$$\text{if } 1 \le k \le r(j) \text{ and } m(j,k) = 2.$$

Until the end of the proof of Lemma 2.3 we omit the subscript j and the superscript j of Γ_j, $P^{j,k}(\cdot)$, $p^{j,k}(\cdot), \ldots$, and "j" of $r(j)$, $m(j,k), \ldots$ and so forth. Namely, we write Γ_j, $P^{j,k}(\cdot)$, $p^{j,k}(\cdot)$, $r(j)$, $m(j,k), \ldots$ as Γ, $P^k(\cdot)$, $p^k(\cdot)$, r, $m(k), \ldots$, respectively. By (2.1) and the factorization theorem we have

$$P(t,\tau,\xi) = P^1(t,\tau,\xi) \circ P^2(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi) + R(t,\tau,\xi) \tag{2.2}$$

for $(t,\xi) \in [0,\delta_1] \times \bar{\Gamma}$ with $|\xi| \ge 1$, where

$$P^k(t,\tau,\xi) = p^k(t,\tau,\xi) + q_0^k(t,\tau,\xi) + q_1^k(t,\tau,\xi) + r^k(t,\tau,\xi),$$

$q_l^k(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(k)-1,-l}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\}))$ $(l = 0,1)$ are positively homogeneous of degree $(m(k) - 1 - l)$ in (τ,ξ) for $|\xi| \ge 1$ and $r^k(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(k)-1,-2}([0,\delta_1] \times$

$(\bar{\Gamma}\backslash\{0\}))$ $(1 \le k \le r)$ and $R(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\}))$ (see, *e.g.*, [5]). Here we denote by $a(t,\tau,\xi) \circ b(t,\tau,\xi)$ the symbol of $a(D_t,D_x)b(t,D_t,D_x)$. For the definition of $\mathscr{S}_{1,0}^{\kappa,\kappa'}(I \times \Gamma)$ we refer to §2 of [11]. Moreover, the $r^k(t,\tau,\xi)$ are classical symbols, *i.e.*, there are symbols $r_l^k(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(k)-1,-2-l}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\}))$ $(l \in \mathbf{Z}_+)$ such that the $r_l^k(t,\tau,\xi)$ are positively homogeneous of degree $(m(k)-3-l)$ in (τ,ξ) for $|\xi| \ge 1$ and

$$r^k(t,\tau,\xi) - \sum_{l=0}^{N-1} r_l^k(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m(k)-1,-2-N}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\})) \quad (N = 1,2,\ldots). \tag{2.3}$$

We write

$$r^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} r_l^k(t,\tau,\xi) \quad \text{in } \mathscr{S}_{1,0}^{m(k)-1,-2}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\}))$$

if (2.3) is valid. We also write

$$\mathscr{S}_{cl}^{m,\nu}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\})) = \{a(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m,\nu}([0,\delta_1] \times (\bar{\Gamma}\backslash\{0\}));$$
$$a(t,\tau,\xi) \text{ is a classical symbol}\}.$$

Define

$$p^k(t,\tau,\xi) = (-1)^{m(k)} p^k(t,-\tau,-\xi),$$
$$q_l^k(t,\tau,\xi) = (-1)^{m(k)-1-l} q_l^k(t,-\tau,-\xi) \quad (l = 0,1)$$
$$\text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((-\bar{\Gamma})\backslash\{0\}).$$

Moreover, we define $r^k(t,\tau,\xi) \in \mathscr{S}_{cl}^{m(k)-1,-2}([0,\delta_1] \times ((-\bar{\Gamma})\backslash\{0\}))$ so that

$$r^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} (-1)^{m(k)-3-l} r_l^k(t,-\tau,-\xi) \tag{2.4}$$
$$\text{in } \mathscr{S}_{1,0}^{m(k)-1,-2}([0,\delta_1] \times ((-\bar{\Gamma})\backslash\{0\})).$$

In fact, we can easily construct a symbol $r^k(t,\tau,\xi)$ for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((-\bar{\Gamma})\backslash\{0\})$ satisfying (2.4). Note that $r^k(t,\tau,\xi)$ is uniquely determined modulo $\mathscr{S}_{1,0}^{m(k)-1,-\infty}([0,\delta_1] \times ((-\bar{\Gamma})\backslash\{0\}))$. Put

$$P^k(t,\tau,\xi) = p^k(t,\tau,\xi) + q_0^k(t,\tau,\xi) + q_1^k(t,\tau,\xi) + r^k(t,\tau,\xi)$$

for $(t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\})$. Then we have the following

Lemma 2.1. *We have*

$$P(t,\tau,\xi) \equiv P^1(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi)$$
$$(\mathrm{mod}\ \mathscr{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\}))).$$

Proof. Write

$$P^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} P_l^k(t,\tau,\xi) \quad \text{in } \mathscr{S}_{1,0}^{m(k)}([0,\delta_1] \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\}))\ \ (1 \le k \le r),$$

where the $P_l^k(t,\tau,\xi)$ are positively homogeneous of degree $(m(k)-l)$ in (τ,ξ). We also write

$$P^1(t,\tau,\xi) \circ P^2(t,\tau,\xi) \circ \cdots \circ P^k(t,\tau,\xi) \sim \sum_{l=0}^{\infty} I_l^{1,2,\dots,k}(t,\tau,\xi)$$
$$\text{in } \mathscr{S}_{1,0}^{m(1)+\cdots+m(k)}([0,\delta_1] \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\}))\ \ (1 \le k \le r),$$

where the $I_l^{1,2,\dots,k}(t,\tau,\xi)$ are positively homogeneous of degree $(m(1)+\cdots+m(k)-l)$. For example, the $I_l^{1,2}(t,\tau,\xi)$ are given by

$$I_l^{1,2}(t,\tau,\xi) = \sum_{\substack{h,\mu,\nu \in \mathbf{Z}_+ \\ h+\mu+\nu=l}} \frac{1}{h!} \partial_\tau^h P_\mu^1(t,\tau,\xi) \cdot D_t^h P_\nu^2(t,\tau,\xi).$$

Then it is easy to see that

$$\begin{aligned} I_l^{1,2}(t,\tau,\xi) &= \sum_{\substack{h,\mu,\nu \in \mathbf{Z}_+ \\ h+\mu+\nu=l}} (-1)^{m(1)+m(2)-l} \frac{1}{h!} (\partial_\tau^h P_\mu^1)(t,-\tau,-\xi)(D_t^h P_\nu^2)(t,-\tau,-\xi) \\ &= (-1)^{m(1)+m(2)-l} I_l^{1,2}(t,-\tau,-\xi) \\ &\qquad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((-\bar{\Gamma})\backslash\{0\}). \end{aligned}$$

Moreover, we can prove by induction on k that

$$I_l^{1,\dots,k}(t,\tau,\xi) = (-1)^{m(1)+\cdots m(k)-l} I_l^{1,\dots,k}(t,-\tau,-\xi) \tag{2.5}$$
$$\text{for } 2 \le k \le r \text{ and } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times ((\bar{\Gamma})\backslash\{0\}).$$

Since $P(t,\tau,\xi)$ is a polynomial of (τ,ξ) and

$$P(t,\tau,\xi) - P^1(t,\tau,\xi) \circ \cdots \circ P^r(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m-1,-\infty}([0,\delta_1] \times (-\bar{\Gamma})\backslash\{0\}),$$

(2.5) proves the lemma. □

We write

$$q_0^k(t,\tau,\xi) = b_0^k(t,\xi)\tau^2 + b_1^k(t,\xi)\tau + b_2^k(t,\xi),$$

$$q_0^k(t,\tau - a_1^k(t,\xi)/3,\xi) = \hat{b}_0^k(t,\xi)\tau^2 + \hat{b}_1^k(t,\xi)\tau + \hat{b}_2^k(t,\xi),$$

if $1 \le k \le r$ and $m(k) = 3$. Then it is obvious that

$$\hat{b}_0^k(t,\xi) = b_0^k(t,\xi)$$

$$\hat{b}_1^k(t,\xi) = b_1^k(t,\xi) - \frac{2}{3}a_1^k(t,\xi)b_0^k(t,\xi),$$

$$\hat{b}_2^k(t,\xi) = b_2^k(t,\xi) + \frac{1}{9}a_1^k(t,\xi)^2 b_0^k(t,\xi) - \frac{1}{3}a_1^k(t,\xi)b_1^k(t,\xi).$$

Let $\mathscr{R}(\xi)$ be a set-valued function, whose values are discrete subsets of $\mathbf{C}$, defined for $\xi \in S^{n-1}$ satisfying the following:

$$\begin{cases} \text{For any } T > 0 \text{ there is } N_T \in \mathbf{Z}_+ \text{ such that} \\ \#\{\lambda \in \mathscr{R}(\xi);\ \mathrm{Re}\,\lambda \in [0,T]\} \le N_T \quad \text{for } \xi \in S^{n-1}. \end{cases}$$

Lemma 2.2. *Assume that $1 \le k \le r$ and $m(k) = 3$. Putting $b(t,\tau,\xi) = \mathrm{sub}\,\sigma(P^k)(t,\tau - a_1^k(t,\xi)/3,\xi)$ we have the following*:

(i) *There is $C_1 > 0$ satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|, 1\right\}|b(t,\tau,\xi)| \le C_1 h_2(t,\tau,\xi;\hat{p}^k)^{1/2} \tag{2.6}$$

$$\text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma} \cap S^{n-1})$$

if and only if there is $C_2 > 0$ satisfying

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|, 1\right\}|b(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| \le C_2 h_2(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi;\hat{p}^k)^{1/2}, \tag{2.7}$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|, 1\right\}|(\partial_\tau b)(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| \le C_2 h_1(t,0,\xi;\hat{p}^k)^{1/2}\ (= \sqrt{2}C_2\hat{a}_2^k(t,\xi)^{1/2}) \tag{2.8}$$

$$\text{for } (t,\xi) \in [0,\delta_1] \times (\overline{\Gamma} \cap S^{n-1}).$$

(ii) (2.6) *is valid if and only if there is* $C_3 > 0$ *satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|b(t,A^k(t,\xi),\xi)| \le C_3 h_2(t,A^k(t,\xi),\xi;\hat{p}^k)^{1/2}, \tag{2.9}$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|(\partial_\tau b)(t,A^k(t,\xi),\xi)| \le C_3 h_1(t,0,\xi;\hat{p}^k)^{1/2}\ (=\sqrt{2}C_3\hat{a}_2^k(t,\xi)^{1/2}) \tag{2.10}$$

for $(t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}\cap S^{n-1})$,

where

$$v^k(t,\xi)=\begin{cases}1 & \text{if } \hat{a}_3^k(t,\xi)\ge 0,\\ -1 & \text{if } \hat{a}_3^k(t,\xi)<0,\end{cases}$$

$$A^k(t,\xi)=v^k(t,\xi)(\hat{a}_2^k(t,\xi)/3)^{1/2}. \tag{2.11}$$

REMARK. Assume that $m(k)=3$. Then we have

$$\begin{aligned} h_2(t,\tau,\xi;\hat{p}^k) &= h_2(t,\tau-a_1^k(t,\xi)/3,\xi;p^k)\\ &= 3\tau^4+\hat{a}_2^k(t,\xi)^2-6\tau\hat{a}_3(t,\xi),\\ h_1(t,\tau,\xi;\hat{p}^k) &= h_1(t,\tau-a_1^k(t,\xi)/3,\xi;p^k)=3\tau^2+2\hat{a}_2^k(t,\xi). \end{aligned} \tag{2.12}$$

Hyperbolicity implies that

$$(\hat{a}_3^k(t,\xi)/2)^2\le(\hat{a}_2^k(t,\xi)/3)^3, \tag{2.13}$$

and the discriminant $\hat{D}^k(t,\xi)$ of $\hat{p}^k(t,\tau,\xi)=0$ in τ is given by

$$\hat{D}^k(t,\xi)\ (=D^k(t,\xi))=4\hat{a}_2^k(t,\xi)^3-27\hat{a}_3^k(t,\xi)^2, \tag{2.14}$$

where $D^k(t,\xi)$ denotes the discriminant of $p^k(t,\tau,\xi)=0$ in τ.

PROOF. Write

$$\hat{p}^k(t,\tau,\xi)=\prod_{l=1}^{3}(\tau-\hat{\lambda}_l^k(t,\xi)),\quad \textit{i.e.},$$

$$\hat{\lambda}_l^k(t,\xi)=\lambda_l^k(t,\xi)+a_1^k(t,\xi)/3\quad(1\le l\le 3).$$

Assume that (2.6) is valid. Then (2.7) is valid with $C_2 \geq C_1$. Fix $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma} \cap S^{n-1})$. We first consider the case where $\hat{\lambda}_l^k(t,\xi) \neq \hat{\lambda}_\mu^k(t,\xi)$ for $1 \leq l < \mu \leq 3$. Then we can write

$$b(t,\tau,\xi) = \sum_{l=1}^{3} b_l(t,\xi)\hat{p}_l^k(t,\tau,\xi), \tag{2.15}$$

where

$$\hat{p}_l^k(t,\tau,\xi) = \prod_{1 \leq \mu \leq m(k),\, \mu \neq l} (\tau - \hat{\lambda}_\mu^k(t,\xi)),$$

$$b_l(t,\xi) = b(t,\hat{\lambda}_l^k(t,\xi),\xi)/\hat{p}_l^k(t,\hat{\lambda}_l^k(t,\xi),\xi) \quad (1 \leq l \leq 3).$$

(2.6) gives

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t-s|, 1\right\}|b_l(t,\xi)| \leq C_1.$$

By (2.15) we have

$$\partial_\tau b(t,\tau,\xi) = \sum_{l=1}^{3} b_l(t,\xi)(2\tau + \hat{\lambda}_l^k(t,\xi)),$$

since $\sum_{\mu=1}^{3} \hat{\lambda}_\mu^k(t,\xi) = 0$. Therefore, we have

$$\begin{aligned} \min\left\{\min_{s \in \mathscr{R}(\xi)} |t-s|, 1\right\}|\partial_\tau b(t,\tau,\xi)| &\leq C_1\left\{6|\tau| + \sqrt{3}\left(\sum_{l=1}^{3} \hat{\lambda}_l^k(t,\xi)^2\right)^{1/2}\right\} \\ &= C_1\{6|\tau| + \sqrt{6}\hat{a}_2^k(t,\xi)^{1/2}\}, \end{aligned} \tag{2.16}$$

since

$$\sum_{l=1}^{3} \hat{\lambda}_l^k(t,\xi)^2 = 2\hat{a}_2^k(t,\xi). \tag{2.17}$$

So (2.13) and (2.16) yield (2.8) with $C_2 \geq (2\sqrt{3}+\sqrt{6})C_1$. Next consider the case where $\hat{\lambda}_1^k(t,\xi) \neq \hat{\lambda}_2^k(t,\xi) = \hat{\lambda}_3^k(t,\xi)$, for instance. Then we have $h_2(t,\hat{\lambda}_2^k(t,\xi),\xi;\hat{p}^k) = 0$ and, therefore, we can write

$$b(t,\tau,\xi) = (\tau - \hat{\lambda}_2^k(t,\xi))\hat{b}(t,\tau,\xi), \tag{2.18}$$

where $\hat{b}(t,\tau,\xi)$ is a linear expression of τ. (2.6) yields

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|\hat{b}(t,\tau,\xi)| \le C_1\{\sqrt{2}|\tau-\hat{\lambda}_1^k(t,\xi)|+|\tau-\hat{\lambda}_2^k(t,\xi)|\}. \tag{2.19}$$

So we have

$$\hat{b}(t,\tau,\xi) = \hat{b}_1(t,\xi)(\tau-\hat{\lambda}_1^k(t,\xi)) + \hat{b}_2(t,\xi)(\tau-\hat{\lambda}_2^k(t,\xi)),$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|\hat{b}_l(t,\xi)| \le \sqrt{2}C_1 \quad (l=1,2), \tag{2.20}$$

where

$$\hat{b}_l(t,\xi) = (-1)^l\hat{b}(t,\hat{\lambda}_{3-l}^k(t,\xi),\xi)/(\hat{\lambda}_1^k(t,\xi)-\hat{\lambda}_2^k(t,\xi)) \quad (l=1,2).$$

Since

$$\partial_\tau b(t,\tau,\xi) = \hat{b}_1(t,\xi)(\tau-\hat{\lambda}_1^k(t,\xi)) + (\hat{b}_1(t,\xi)+2\hat{b}_2(t,\xi))(\tau-\hat{\lambda}_2^k(t,\xi)), \tag{2.21}$$

(2.13) and (2.17)–(2.21) give

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|(\partial_\tau b)(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)|$$
$$\le 4\sqrt{2}C_1\{(\hat{a}_2^k(t,\xi)/3)^{1/2}+2\hat{a}_2^k(t,\xi)^{1/2}\} \le 12\sqrt{2}C_1\hat{a}_2^k(t,\xi)^{1/2},$$

which proves that (2.8) is valid. Finally consider the case where $\hat{\lambda}_1^k(t,\xi) = \hat{\lambda}_2^k(t,\xi) = \hat{\lambda}_3^k(t,\xi)$ $(=0)$. Then we have $\hat{a}_2^k(t,\xi) = \hat{a}_3^k(t,\xi) = 0$,

$$h_2(t,\tau,\xi;\hat{p}^k) = 3\tau^4 \quad \text{and} \quad h_1(t,\tau,\xi;\hat{p}^k) = 3\tau^2.$$

Therefore, we can write

$$b(t,\tau,\xi) = \tau^2\hat{b}(t,\xi), \tag{2.22}$$

where

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|\hat{b}(t,\xi)| \le \sqrt{3}C_1. \tag{2.23}$$

This yields

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|(\partial_\tau b)(t,(\hat{a}_3^k(t,\xi)/2)^{1/3},\xi)| = 0 \le h_1(t,0,\xi;\hat{p}^k)^{1/2} \ (=0).$$

Next we assume that (2.7) and (2.8) are valid. Write

$$b(t,\tau,\xi) = b(t,(\hat{a}_3^k/2)^{1/3},\xi) + (\partial_\tau b)(t,(\hat{a}_3^k/2)^{1/3},\xi)(\tau - (\hat{a}_3^k/2)^{1/3}) + \frac{1}{2}(\partial_\tau^2 b)(t,0,\xi)(\tau - (\hat{a}_3^k/2)^{1/3})^2,$$

(2.24) $$h_2(t,\tau,\xi;\hat{p}^k) = 9((\hat{a}_2^k/3)^2 - (\hat{a}_3^k/2)^{4/3}) + 6(\hat{a}_3^k/2)^{2/3}(\tau - (\hat{a}_3^k/2)^{1/3})^2 + 3(\tau^2 - (\hat{a}_3^k/2)^{2/3})^2,$$

where $\hat{a}_l^k = \hat{a}_l^k(t,\xi)$ $(l = 2,3)$. Since

(2.25) $$h_2(t,(\hat{a}_3^k/2)^{1/3},\xi;\hat{p}^k) = 9((\hat{a}_2^k/3)^2 - (\hat{a}_3^k/2)^{4/3}),$$

we have

(2.26) $$h_2(t,(\hat{a}_3^k/2)^{1/3},\xi;\hat{p}^k) \le h_2(t,\tau,\xi;\hat{p}^k).$$

Moreover, we have

(2.27) $$\begin{aligned}(\tau - (\hat{a}_3^k/2)^{1/3})^2 &\le |\tau^2 - (\hat{a}_3^k/2)^{2/3}| + 2|(\hat{a}_3^k/2)^{1/3}(\tau - (\hat{a}_3^k/2)^{1/3})| \\ &\le h_2(t,\tau,\xi;\hat{p}^k)^{1/2}/\sqrt{3} + 2h_2(t,\tau,\xi;\hat{p}^k)^{1/2}/\sqrt{6} \\ &\le 2h_2(t,\tau,\xi;\hat{p}^k)^{1/2},\end{aligned}$$

(2.28) $$\begin{aligned}\{(\hat{a}_2^k)^{1/2}(\tau - (\hat{a}_3^k/2)^{1/3})\}^4 &= h_2(t,(\hat{a}_3^k/2)^{1/3},\xi;\hat{p}^k)(\tau - (\hat{a}_3^k/2)^{1/3})^4 \\ &\quad + \{3(\hat{a}_3^k/2)^{2/3}(\tau - (\hat{a}_3^k/2)^{1/3})^2\}^2 \\ &\le 5h_2(t,\tau,\xi;\hat{p}^k)^2.\end{aligned}$$

We may assume that $|(\partial_\tau^2 b)(t,0,\xi)| \le C_2$. Therefore, (2.6) is valid with $C_1 \ge 6C_2$, which proves the assertion (i). (2.25) gives

$$\begin{aligned}h_2(t,(\hat{a}_3^k/2)^{1/3},\xi;\hat{p}^k) &= 9\{(\hat{a}_2^k/3)^2 - (\hat{a}_3^k/2)^{4/3}\} \\ &= 9\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\}\{(\hat{a}_2^k/3)^{1/2} + (|\hat{a}_3^k|/2)^{1/3}\} \\ &\quad \times \{(\hat{a}_2^k/3) + (|\hat{a}_3^k|/2)^{2/3}\}.\end{aligned}$$

So we have

(2.29) $$9(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \le h_2(t,(\hat{a}_3^k/2)^{1/3},\xi;\hat{p}^k).$$

On the other hand, we have

$$
\begin{aligned}
h_2(t, A^k(t,\xi), \xi; \hat{p}^k) &= 12(\hat{a}_2^k/3)^2 - 12(\hat{a}_2^k/3)^{1/2}(|\hat{a}_3^k|/2) \\
&= 12(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \\
&\quad \times \{(\hat{a}_2^k/3) + (\hat{a}_2^k/3)^{1/2}(|\hat{a}_3^k|/2)^{1/3} + (|\hat{a}_3^k|/2)^{2/3}\}.
\end{aligned}
$$

By (2.13) we have

$$
\begin{aligned}
(2.30)\qquad & 12(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \\
&\le h_2(t, A^k(t,\xi), \xi; \hat{p}^k) \le 36(\hat{a}_2^k/3)^{3/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\},
\end{aligned}
$$

which, with (2.26) and (2.29), yields

$$
(2.31)\qquad h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k) \le h_2(t, A^k(t,\xi), \xi; \hat{p}^k) \le 4h_2(t, (\hat{a}_3^k/2)^{1/3}, \xi; \hat{p}^k).
$$

Now we can prove the assertion (ii). Note that

$$
\partial_t\partial_\tau^2 p^k(t,\tau,\xi) = 2\partial_t a_1^k(t,\xi),
$$

$$
(2.32)\qquad \partial_\tau b(t,\tau,\xi) = 2b_0^k(t,\xi)\tau + \hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi).
$$

So we have

$$
\begin{aligned}
(2.33)\qquad & |(\partial_\tau b)(t, A^k(t,\xi), \xi) - (\partial_\tau b)(t, (\hat{a}_3^k(t,\xi)/2)^{1/3}, \xi)| \\
&= 2|b_0(t,\xi)|\{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\} \\
&\le 2|b_0(t,\xi)|(\hat{a}_2^k(t,\xi)/3)^{1/2}
\end{aligned}
$$

since $|A^k(t,\xi) - (\hat{a}_3^k(t,\xi)/2)^{1/3}| = (\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}$. This implies that (2.8) is valid if and only if (2.10) is valid. We have also

$$
\begin{aligned}
(2.34)\qquad & |b(t, A^k(t,\xi), \xi) - b(t, (\hat{a}_3^k(t,\xi)/2)^{1/3}, \xi)| \\
&\le \{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\}|(\partial_\tau b)(t, (\hat{a}_3^k(t,\xi)/2)^{1/3}, \xi)| \\
&\quad + \{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\}^2|(\partial_\tau^2 b)(t,0,\xi)|/2.
\end{aligned}
$$

It follows from (2.25) and (2.26) that

$$
\begin{aligned}
(2.35)\qquad & 3(\hat{a}_2^k(t,\xi)/3)^{1/2}\{(\hat{a}_2^k(t,\xi)/3)^{1/2} - (|\hat{a}_3^k(t,\xi)|/2)^{1/3}\} \\
&\le h_2(t, (\hat{a}_3^k(t,\xi)/2)^{1/3}, \xi; \hat{p}^k)^{1/2} \le h_2(t, A^k(t,\xi), \xi; \hat{p}^k)^{1/2},
\end{aligned}
$$

since $(\alpha-\beta)^{1/2} \geq \alpha^{1/2}-\beta^{1/2}$ if $\alpha \geq \beta \geq 0$. This, together with (2.8) and (2.34), proves that (2.7) and (2.8) hold if and only if (2.9) and (2.10) hold. □

LEMMA 2.3. *Assume that* $1 \leq k \leq r$, *and* $m(k)=2$. *Then there is* $C_1>0$ *satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|sub\ \sigma(P^k)(t,\tau,\xi)| \leq C_1 h_1(t,\tau,\xi;p^k)^{1/2} \tag{2.36}$$

$$\textit{for}\ \ (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\bar{\Gamma}\cap S^{n-1})$$

if and only if there is $C_2>0$ *satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|sub\ \sigma(P^k)(t,-a_1^k(t,\xi)/2,\xi)| \tag{2.37}$$

$$\leq C_2 h_1(t,-a_1^k(t,\xi)/2,\xi;p^k)^{1/2} \quad \textit{for}\ \ (t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}\cap S^{n-1}).$$

REMARK. Assume that $m(k)=2$. Then we have

$$h_1(t,\tau,\xi;p^k) = 2(\tau+a_1^k(t,\xi)/2)^2 + 2\hat{a}_2^k(t,\xi). \tag{2.38}$$

PROOF. We have

$$sub\ \sigma(P^k)(t,\tau,\xi) = sub\ \sigma(P^k)(t,-a_1^k(t,\xi)/2,\xi)$$
$$+ (\partial_\tau\ sub\ \sigma(P^k))(t,0,\xi)(\tau+a_1^k(t,\xi)/2).$$

Therefore, this, together with (2.38), proves the lemma. □

Define

$$\beta^k(t,\xi) = sub\ \sigma(P^k)(t,A^k(t,\xi)-a_1^k(t,\xi)/3,\xi) \tag{2.39}$$

for $1 \leq k \leq r$ with $m(k)=3$. We note that

$$\partial_t a_2^k(t,\xi) = 2a_1^k(t,\xi)\partial_t a_1^k(t,\xi)/3 - \partial_t\hat{a}_2^k(t,\xi),$$
$$sub\ \sigma(P^k)(t,\tau-a_1^k(t,\xi)/3,\xi)$$
$$= q_0^k(t,\tau-a_1^k(t,\xi)/3,\xi) + i\partial_t a_1^k(t,\xi)\cdot\tau - \frac{i}{2}\partial_t\hat{a}_2^k(t,\xi)$$

if $1 \leq k \leq r$ and $m(k)=3$.

LEMMA 2.4. *Let* $k \in \mathbf{N}$ *satisfy* $1 \le k \le r$ *and* $m(k) = 3$. (i) *Assume that* $\hat{a}_2^k(t,\xi) \not\equiv 0$ *in* (t,ξ). *Then there is* $C_1 > 0$ *satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|sub\ \sigma(P^k)(t,\tau - a_1^k(t,\xi)/3,\xi)| \le C_1 h_2(t,\tau,\xi;\hat{p}^k)^{1/2} \quad for\ (t,\tau,\xi) \in [0,\delta_1]\times\mathbf{R}\times(\bar{\Gamma}\cap S^{n-1}) \tag{2.40}$$

if and only if there is $C_2 > 0$ *satisfying*

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|\beta^k(t,\xi)|\hat{a}_2^k(t,\xi) \le C_2(\hat{D}^k(t,\xi)\hat{a}_2^k(t,\xi))^{1/2}, \tag{2.41}$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|,1\right\}|\hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi)| \le C_2\hat{a}_2^k(t,\xi)^{1/2} \quad for\ (t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}\cap S^{n-1}). \tag{2.42}$$

(ii) *Assume that* $\hat{a}_2^k(t,\xi) \equiv 0$. *Then* (2.40) *is valid if and only if*

$$\hat{b}_1^k(t,\xi) + i\partial_t a_1^k(t,\xi) = \hat{b}_2^k(t,\xi) = 0 \quad for\ (t,\xi)\in[0,\delta_1]\times(\bar{\Gamma}\cap S^{n-1}).$$

PROOF. Assume that $\hat{a}_2(t,\xi) \not\equiv 0$ in (t,ξ). By virtue of Lemma 2.2 it is enough to prove that the conditions (2.9) and (2.10) are equivalent to the conditions (2.41) and (2.42). Here we may modify the constants appropriately. Since $|A^k(t,\xi)| = (\hat{a}_2^k(t,\xi)/3)^{1/2}$, by (2.32) we see that (2.10) is valid if and only if (2.42) is valid. (2.13) and (2.14) yield

$$\begin{aligned} &108(\hat{a}_2^k/3)^2\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\} \\ &\le \hat{D}^k(t,\xi) = 108\{(\hat{a}_2^k/3)^3 - (\hat{a}_3^k/2)^2\} = 108\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\} \\ &\quad\times\{(\hat{a}_2^k/3)^2 + (\hat{a}_2^k/3)(|\hat{a}_3^k|/2)^{2/3} + (|\hat{a}_3^k|/2)^{4/3}\} \\ &\le 324(\hat{a}_2^k/3)^2\{(\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}\}, \end{aligned} \tag{2.43}$$

where $\hat{a}_l^k = \hat{a}_l^k(t,\xi)$ $(l = 2,3)$. This, together with (2.30), yields

$$3(\hat{a}_2^k/3)h_2(t,A^k(t,\xi),\xi;\hat{p}^k) \le \hat{D}^k(t,\xi) \le 54(\hat{a}_2^k/3)h_2(t,A^k(t,\xi),\xi;\hat{p}^k), \tag{2.44}$$

since

$$(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\} \le (\hat{a}_2^k/3) - (|\hat{a}_3^k|/2)^{2/3}$$
$$\le 2(\hat{a}_2^k/3)^{1/2}\{(\hat{a}_2^k/3)^{1/2} - (|\hat{a}_3^k|/2)^{1/3}\}.$$

Therefore, if (2.9) is valid, then (2.41) is valid with $C_2 = \sqrt{3}C_3$. Applying the Weierstrass preparation theorem to $\hat{a}_2^k(t,\xi)$, we can prove that (2.9) is valid with $C_3 = 6C_2$ if (2.41) is valid, which proves the assertion (i). Next assume that $\hat{a}_2^k(t,\xi) \equiv 0$ in (t,ξ). Then, by (2.12) and (2.13) we have

$$\hat{a}_3(t,\xi) \equiv \hat{D}^k(t,\xi) \equiv 0 \quad \text{and} \quad h_2(t,\tau,\xi;\hat{p}^k) \equiv 3\tau^4,$$

which proves the assertion (ii). □

For $(t_0,x^0) \in (0,\delta_1] \times \mathbf{R}^n$ and $\varepsilon > 0$ we put

$$\Omega_\varepsilon(t_0,x^0) = \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\ t_0 - t > \varepsilon|x - x^0|^2\}.$$

Lemma 2.5. *Assume that the Cauchy problem* (CP) *is C^∞ well-posed and has finite propagation property. Then there is $\varepsilon_0 > 0$ such that for $(t_0,x^0) \in [0,\infty) \times \mathbf{R}^n$ and $p \in \mathbf{Z}_+$ there are $C > 0$ and $q \in \mathbf{Z}_+$ satisfying*

$$|u|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} \le C|Pu|_{q,\Omega_{\varepsilon_0}(t_0,x^0)}$$

for any $u \in C^\infty(\mathbf{R}^{n+1})$ with $u(t,x)|_{t<0} = 0$. Here $|f|_{p,K}$ is defined by

$$|f|_{p,K} = \sup_{(t,x)\in K,\, j+|\alpha|\le p} |D_t^j D_x^\alpha f(t,x)|.$$

Proof. We can choose $\varepsilon_0 > 0$ so that

$$(\{(t_1,x^1)\} - \Gamma_0) \cap \{t \ge 0\} \subset \Omega_{\varepsilon_0}(t_0,x^0)$$

if $(t_0,x^0) \in (0,\delta_1] \times \mathbf{R}^n$, $(t_1,x^1) \in \Omega_\varepsilon(t_0,x^0)$ and $t_1 \ge 0$. Here Γ_0 is a proper convex closed cone in $\mathbf{R}^{n+1}$ such that $\Gamma_0 \subset \{t > 0\} \cup \{0\}$ and Γ_0 satisfies the following:

$$u(t,x) = 0 \quad \text{in } \Gamma_0(t_0,x^0)\ (\equiv \{(t_0,x^0)\} - \Gamma_0)$$
$$\text{if } (t_0,x^0) \in [0,\delta_1] \times \mathbf{R}^n,\ u \in C^\infty(\mathbf{R}^{n+1}),$$
$$\operatorname{supp} u \subset \{t \ge 0\} \text{ and } P(t,D_t,D_x)u(t,x) = 0 \text{ in } \Gamma_0(t_0,x^0).$$

Define

$$X = \{f \in C^\infty(\mathbf{R}^{n+1});\ \mathrm{supp}\, f \subset \{t \geq 0\}\}.$$

X is a closed subspace of the Fréchet space $C^\infty(\mathbf{R}^{n+1})$. The operator $X \ni f(t,x) \mapsto u(t,x) \in X$ is a closed operator, where $u(t,x)$ is a unique solution in X satisfying $Pu(t,x) = f(t,x)$. So Banach's closed graph theorem proves that for any compact subset K of $[0,\infty) \times \mathbf{R}^n$ and $p \in \mathbf{Z}_+$, there are a compact subset K' of $[0,\infty) \times \mathbf{R}^n$, $C_{p,K} > 0$ and $q \in \mathbf{Z}_+$ satisfying

$$|u|_{p,K} \leq C_{p,K}|Pu|_{q,K'} \quad \text{for } u \in X. \tag{2.45}$$

It follows from [2] that for any $u \in X$ and $(t_0, x^0) \in [0, \delta_1] \times \mathbf{R}^n$ there are $f \in X$ and $C' > 0$ such that $f = Pu$ in $\Omega_{\varepsilon_0}(t_0, x^0)$ and

$$|f|_{q,\mathbf{R}^{n+1}} \leq C'|Pu|_{q,\Omega_{\varepsilon_0}(t_0,x^0)} \tag{2.46}$$

(see, also, [6]). By the assumptions there is $v \in X$ satisfying $Pv = f$. Then finite propagation property implies that $v(t,x) = u(t,x)$ in $\Omega_{\varepsilon_0}(t_0, x^0)$. (2.45) with $K = \overline{\Omega_{\varepsilon_0}(t_0,x^0)} \cap \{t \geq 0\}$ and (2.46) yield

$$\begin{aligned} |u|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} &= |v|_{p,\Omega_{\varepsilon_0}(t_0,x^0)} \leq C_{p,K}|f|_{q,K'} \leq C_{p,K}|f|_{q,\mathbf{R}^{n+1}} \\ &\leq C'C_{p,K}|Pu|_{p,\Omega_{\varepsilon_0}(t_0,x^0)}, \end{aligned}$$

which proves the lemma. □

3. The Triple Characteristic Factors

We factorized $p(t,\tau,\xi)$ as (2.1):

$$p(t,\tau,\xi) = \prod_{k=1}^{r(j)} p^{j,k}(t,\tau,\xi) \quad \text{for } (t,\tau,\xi) \in [0,\delta_1] \times \mathbf{R} \times (\overline{\Gamma}_j \cap S^{n-1}),$$

where $1 \leq j \leq N_0$. In this section we omit the subscript j and the superscript j, and "j" of $r(j)$ and $m(j,k)$, in the same manner as in §2. Fix $k_0 \in \mathbf{N}$ so that $1 \leq k_0 \leq r$ and $m(k_0) = 3$. We also define $D_l^{k_0}(t,\xi)$ $(0 \leq l \leq 3)$ by

$$\tau^3 + \sum_{l=1}^{3} D_l^{k_0}(t,\xi)\tau^{3-l} = \prod_{1 \leq k < l \leq 3} (\tau + \mu_{k,l}^{k_0}(t,\xi)),$$

$$D_0^{k_0}(t,\xi) \equiv 1,$$

where $\mu_{k,l}^{k_0}(t,\xi) = (\lambda_k^{k_0}(t,\xi) - \lambda_l^{k_0}(t,\xi))^2$. Then we have

$$D_3^{k_0}(t,\xi) = \hat{D}^{k_0}(t,\xi) = 4\hat{a}_2^{k_0}(t,\xi)^3 - 27\hat{a}_3^{k_0}(t,\xi)^2,$$

$$D_2^{k_0}(t,\xi) = 9\hat{a}_2^{k_0}(t,\xi)^2,$$

$$D_1^{k_0}(t,\xi) = 6\hat{a}_2^{k_0}(t,\xi).$$

By the factorization theorem we can write

$$\begin{aligned}(3.1)\qquad P(t,\tau,\xi) = {} & P^1(t,\tau,\xi)\circ\cdots\circ P^{k_0-1}(t,\tau,\xi)\circ P^{k_0+1}(t,\tau,\xi)\\ & \circ\cdots\circ P^r(t,\tau,\xi)\circ P^{k_0}(t,\tau,\xi) + R(t,\tau,\xi),\end{aligned}$$

where $R(t,\tau,\xi) \in \mathscr{S}_{1,0}^{m-1,-\infty}([0,\delta_1]\times(\bar{\Gamma}\backslash\{0\}))$. We note that the $P^k(t,\tau,\xi)$ are different from the $P^k(t,\tau,\xi)$ in (2.2) if $k_0 \neq r$, and that whether (2.18) and (2.19) in Lemma 2.5 of [11] are satisfied or not does not depend on the order of the product in (2.2) (see Lemma 2.5 and its remark of [11]). We may assume that $P^k(t,\tau,\xi)$ are defined for $(t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times((\bar{\Gamma}\cup(-\bar{\Gamma}))\backslash\{0\})$ as stated in §2. For $(t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times((-\bar{\Gamma})\backslash\{0\})$ we define $R(t,\tau,\xi)$ by

$$\begin{aligned}R(t,\tau,\xi) = {} & P(t,\tau,\xi) - P^1(t,\tau,\xi)\circ\cdots\circ P^{k_0-1}(t,\tau,\xi)\circ P^{k_0+1}(t,\tau,\xi)\\ & \circ\cdots\circ P^r(t,\tau,\xi)\circ P^{k_0}(t,\tau,\xi)\end{aligned}$$

(see Lemma 2.1). Now fix k_0, and write $P^{k_0}(t,\tau,\xi)$, $p^{k_0}(t,\tau,\xi)$, $D_l^{k_0}(t,\xi),\dots$ as $P(t,\tau,\xi)$, $p(t,\tau,\xi)$, $D_l(t,\xi),\dots$, *i.e.*,

$$p(t,\tau,\xi) = \tau^3 + a_1(t,\xi)\tau^2 + a_2(t,\xi)\tau + a_3(t,\xi),$$

$$\hat{p}(t,\tau,\xi) = p(t,\tau - a_1(t,\xi)/3,\xi) = \tau^3 - \hat{a}_2(t,\xi)\tau + \hat{a}_3(t,\xi),$$

$$P(t,\tau,\xi) = p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi) + r(t,\tau,\xi)$$

until Lemma 3.5, where $q_l(t,\tau,\xi)\in\mathscr{S}_{1,0}^{2,-l}([0,\delta_1]\times(\bar{\Gamma}\backslash\{0\}))$ is positively homogeneous of degree $(2-l)$ in (τ,ξ) for $|\xi|\geq 1$ $(l=0,1)$ and $r(t,\tau,\xi)\in\mathscr{S}_{1,0}^{2,-2}([0,\delta_1]\times(\bar{\Gamma}\backslash\{0\}))$. Let $t_0\in[0,\delta_1/2]$, $\xi^0\in\Gamma\cap S^{n-1}$ and $\theta_0>0$, and let $T(\theta), \Xi_l(\theta)\in C^\infty((0,\theta_0])\cap C([0,\theta_0])$ $(1\leq l\leq n)$ be real-valued functions satisfying the following:

(i) $0 < t_0 + T(\theta) \leq \delta_1$ for $\theta\in(0,\theta_0]$.
(ii) $T(0)=0$ and $\Xi(0)=\xi^0$, where $\Xi(\theta) = (\Xi_1(\theta),\dots,\Xi_n(\theta))$.
(iii) $\Xi(\theta)\in S^{n-1}$ for $\theta\in[0,\theta_0]$ and the $\Xi_l(\theta)$ are real analytic in $[0,\theta_0]$.
(iv) $T(\theta)$ can be expanded into a convergent Puiseux series of $\theta\in[0,\theta_0]$.

We say that $T(\theta)$ and $\Xi(\theta)$ satisfy the condition (T,Ξ) if the above conditions (i)–(iv) are satisfied.

(I) The case where $D_3(t,\Xi(\theta)) \not\equiv 0$ in (t,θ).

Applying the Weierstrass preparation theorem, we can write

$$D_3(t_0+t,\Xi(\theta))\hat{a}_2(t_0+t,\Xi(\theta))$$
$$= \sum_{l=l_0}^{\infty} d_l(t)\theta^l = \theta^{l_0} d(t,\theta)\prod_{i=1}^{n_0}(t-t_i(\theta)), \quad d(t,\theta)\neq 0$$

for $(t,\theta)\in[-\delta_0,\delta_0]\times[0,\theta_0]$, where $0<\delta_0\le\delta_1-t_0$, $d_{l_0}(t)\not\equiv 0$ and $t_i(\theta)\equiv t_i(\theta;t_0,\Xi)$. The $t_i(\theta)$ can be expanded into a convergent Puiseux series of θ in $[0,\theta_0]$, with a modification of θ_0 if necessary. Put

$$\mathscr{R}_0(\Xi(\theta);p) = \{t_0+t_i(\theta);\ 1\le i\le n_0\},$$
$$\tilde{\mathscr{R}}_0(\Xi(\theta);p) = \{(t_0+\mathrm{Re}\ t_i(\theta))_+;\ 1\le i\le n_0\}.$$

Then we have

$$\mathscr{R}_0(\Xi(\theta)) \supset \tilde{\mathscr{R}}_0(\Xi(\theta);p) \quad (\theta\in(0,\theta_0]),$$
$$\min_{s\in\mathscr{R}_0(\Xi(\theta))}|t_0+T(\theta)-s| \le \min_{s\in\tilde{\mathscr{R}}_0(\Xi(\theta);p)}|t_0+T(\theta)-s| \tag{3.2}$$
$$\le \min_{s\in\mathscr{R}_0(\Xi(\theta);p)}|t_0+T(\theta)-s| \quad (\theta\in(0,\theta_0]).$$

(2.43) implies that $D_3(t,\xi)=0$ if $D_3(t,\xi)\hat{a}_2(t,\xi)=0$.

(II) The case where $D_3(t,\Xi(\theta))\equiv 0$ and $\hat{a}_2(t,\Xi(\theta))\not\equiv 0$ in (t,θ).

Similarly, we can write

$$\hat{a}_2(t_0+t,\Xi(\theta)) = \theta^{l_0} d(t,\theta)\prod_{i=1}^{n_0}(t-t_i(\theta)), \quad d(t,\theta)\neq 0$$
$$\text{for } (t,\theta)\in[-\delta_0,\delta_0]\times[0,\theta_0],$$

where $t_i(\theta)\equiv t_i(\theta;t_0,\Xi)$ is expanded into a convergent Puiseux series of θ in $[0,\theta_0]$, with modifications of θ_0 and δ_0 if necessary. Since $D_2(t,\xi)=9\hat{a}_2(t,\xi)^2$, we have also

$$\mathscr{R}_0(\Xi(\theta)) \supset \{(t_0+\mathrm{Re}\ t_i(\theta))_+;\ 1\le i\le n_0\}\ (\equiv \tilde{\mathscr{R}}_0(\Xi(\theta);p)) \quad (\theta\in(0,\theta_0]).$$

Putting $\mathscr{R}_0(\Xi(\theta);p)=\{t_0+t_i(\theta);\ 1\le i\le n_0\}$, we have (3.2).

(III) The case where $\hat{a}_2(t,\Xi(\theta))\equiv 0$ in (t,θ).

By (2.13) we have $\hat{p}(t, \tau, \Xi(\theta)) = \tau^3$ and put

$$\mathscr{R}_0(\Xi(\theta); p) = \tilde{\mathscr{R}}_0(\Xi(\theta); p) = \varnothing \ (\subset \mathscr{R}_0(\Xi(\theta))),$$

$n_0 = 0$ and $l_0 = \infty$.

Now we define

$$\hat{\mu} \ (\equiv \hat{\mu}(t_0, \xi^0, T, \Xi)) = \{\mathrm{Ord}_{\theta \downarrow 0}\, \hat{a}_2(t_0 + T(\theta), \Xi(\theta))\}/2,$$

$$\hat{\mu}_0 \ (\equiv \hat{\mu}_0(t_0, \xi^0, T, \Xi)) = \{\mathrm{Ord}_{\theta \downarrow 0}\, D_3(t_0 + T(\theta), \Xi(\theta))\}/2 - \hat{\mu},$$

$$\mu_1 \ (\equiv \mu_1(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta \downarrow 0}\left\{ \min_{s \in \mathscr{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \alpha(t_0 + T(\theta), \Xi(\theta)) \right\},$$

$$\mu_2 \ (\equiv \mu_2(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta \downarrow 0}\left\{ \min_{s \in \mathscr{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \beta(t_0 + T(\theta), \Xi(\theta)) \right\},$$

$$\mu_3 \ (\equiv \mu_3(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta \downarrow 0}\left\{ \min_{s \in \mathscr{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s|^2 \hat{c}_1(t_0 + T(\theta), \Xi(\theta)) \right\},$$

$$\delta \ (\equiv \delta(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta \downarrow 0}\left\{ \min_{s \in \mathscr{R}_0(\Xi(\theta); p)} |t_0 + T(\theta) - s| \right\},$$

where

$$\alpha(t, \xi) = \hat{b}_1(t, \xi) + i\partial_t a_1(t, \xi),$$

$$\hat{c}_1(t, \xi) = sub^2\, \sigma(P)(t, -a_1(t, \xi)/3, \xi)$$

and $\beta(t, \xi)$ is defined by (2.39) with $k = k_0$, and $\hat{\mu} = \hat{\mu}_0 = \hat{\mu}_0 - \hat{\mu} = \infty$ and $\delta = 0$ in the case (III). Here for $f \in C([0, \theta_0])$ $\mathrm{Ord}_{\theta \downarrow 0} f(\theta) = v$ $(\in \mathbf{R})$ means that there is $c \in \mathbf{C} \setminus \{0\}$ satisfying $f(\theta) = c\theta^v(1 + o(1))$ as $\theta \downarrow 0$. We write $\mathrm{Ord}_{\theta \downarrow 0} f(\theta) = \infty$ if $f(\theta) = O(\theta^N)$ as $\theta \downarrow 0$ for any $N \in \mathbf{Z}_+$. Note that

$$(\partial_\tau\, sub\, \sigma(P))(t, A(t, \xi) - a_1(t, \xi)/3, \xi) = 2b_0(t, \xi)A(t, \xi) + \alpha(t, \xi). \tag{3.3}$$

It follows from (2.43) and (2.44) that

$$\hat{\mu}_0 \geq 2\hat{\mu},$$

$$\hat{\mu}_0 = \mathrm{Ord}_{\theta \downarrow 0}\, h_2(t_0 + T(\theta), A(t_0 + T(\theta), \Xi(\theta)), \Xi(\theta); \hat{p})^{1/2}.$$

Proposition 3.1. *If*

$$\min\{\mu_1, \mu_3\} < \hat{\mu} \quad \text{or} \quad \mu_2 < \hat{\mu}_0, \tag{3.4}$$

then "the Cauchy problem (CP) *is not* C^∞ *well-posed" or "*(CP) *does not have finite propagation property."*

REMARK. When one replaces $\mathscr{R}_0(\Xi(\theta);p)$ by $\tilde{\mathscr{R}}_0(\Xi(\theta);p)$ in the definitions of μ_k $(k=1,2,3)$ and δ, one can show that the proposition is valid, using (3.2). It follows from (2.5) and (2.26) of [11] that whether (3.4) holds or not does not depend on the order of the product in (2.2), although the μ_k are defined under the factorization (3.1). Indeed, if $1 \le k \le r$, $m(k)=3$ and $a(t,\tau,\xi)$ is a polynomial of τ satisfying $a(t,\tau,\xi) = O(h_{m-1}(t,\tau,\xi)^{1/2})$ for $(t,\tau,\xi) \in [0,\delta_1] \times I_k \times (\bar{\Gamma} \cap S^{n-1})$, then there are $b_\mu(t,\xi)$ $(1 \le \mu \le 3)$ and $C>0$ such that

$$a(t,\tau,\xi) = \sum_{\mu=1}^{3} b_\mu(t,\xi) p_\mu^k(t,\tau,\xi),$$

$$|b_\mu(t,\xi)| \le C \quad (1 \le \mu \le 3).$$

So we have

$$|\partial_\tau a(t,\tau,\xi)| \le C \sum_{\mu=1}^{3} |\partial_\tau p_\mu^k(t,\tau,\xi)| \le C' h_1(t,\tau,\xi;p^k)^{1/2}.$$

COROLLARY 3.2. *Assume that the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* $(t_0,\xi^0) \in [0,\delta_1/2] \times (\Gamma \cap S^{n-1})$. *Then we have*

$$\hat{\mu}_0(t_0,\xi^0,T,\Xi) \le \mu_2(t_0,\xi^0,T,\Xi),$$

$$\hat{\mu}(t_0,\xi^0,T,\Xi) \le \mu_k(t_0,\xi^0,T,\Xi) \quad (k=1,3)$$

if $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T,Ξ).

REMARK. The corollary does not depend on the order of the product in (2.2).

In the rest of this section we shall prove Proposition 3.1, and give several lemmas. Assume that (3.4) is satisfied. Then we have $\delta < \infty$ since $\mu_k \ge \delta$ $(k=1,2)$ and $\mu_3 \ge 2\delta$. Moreover, we have $\hat{\mu}_0 > 0$ and $D_3(t_0,\xi^0) = 0$. There is $c_0 > 0$ such that

$$\min_{s \in \mathscr{R}_0(\Xi(\theta);p)} |t_0 + T(\theta) - s| \ge c_0 \theta^\delta \quad \text{for } \theta \in [0,\theta_0].$$

In the case (III) we may take $c_0 = 1$ and $n_0 = 0$. For $v \in \mathbf{R}$ we put

$$T_v(\theta) = T(\theta) + v\theta^\delta. \tag{3.5}$$

In the cases (I) and (II) we have

$$\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{2\hat{\mu}}(d(v) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $d(v) \not\equiv 0$ is a polynomial of v with real coefficients. It is easy to see that

$$d(v) > 0 \quad \text{for } v \in [-c_0/2, c_0/2],$$

$$\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta))^{1/2} = \theta^{\hat{\mu}}(\sqrt{d(v)} + o(1))$$
$$\text{uniformly in } v \in [-c_0/2, c_0/2] \text{ as } \theta \downarrow 0.$$

Write

$$\alpha(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_1 - \delta}(d_1(v) + o(1)) \quad \text{as } \theta \downarrow 0 \quad \text{if } \alpha(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

$$\beta(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_2 - \delta}(d_2(v) + o(1)) \quad \text{as } \theta \downarrow 0 \quad \text{if } \beta(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

$$\hat{c}_1(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_3 - 2\delta}(d_3(v) + o(1)) \quad \text{as } \theta \downarrow 0 \quad \text{if } \hat{c}_1(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

where $\tilde{\mu}_k \in \mathbf{Q}$ and the $d_k(v)$ ($\not\equiv 0$) are polynomials of v. Here, for instance, we put $\tilde{\mu}_2 = \infty$ if $\beta(t, \Xi(\theta)) \equiv 0$ in (t, θ). We note that $\tilde{\mu}_l \le \mu_l$ $(1 \le l \le 3)$. It is easy to see that

$$\{\mathrm{Ord}_{\theta\downarrow 0}\, D_3(t_0 + T_v(\theta), \Xi(\theta))\}/2 - \hat{\mu} = \hat{\mu}_0 \quad \text{for } v \in [-c_0/2, c_0/2]$$

in the case (I). We also write

$$\hat{a}_3(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\mu_4}(d_4(v) + o(1)) \quad \text{as } \theta \downarrow 0 \quad \text{if } \hat{a}_3(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

where $d_4(v)$ ($\not\equiv 0$) is a polynomial of v with real coefficients. Therefore, there are $v_0 \in (c_0/4, c_0/2)$ and $s_0 > 0$ such that $I_0 \equiv [v_0 - s_0, v_0 + s_0] \subset [c_0/4, c_0/2]$ and

$$d_1(v) \ne 0 \quad \text{if } \alpha(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

$$d_2(v) \ne 0 \quad \text{if } \beta(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

$$d_3(v) \ne 0 \quad \text{if } \hat{c}_1(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta),$$

$$d_4(v) \ne 0 \quad \text{if } \hat{a}_3(t, \Xi(\theta)) \not\equiv 0 \text{ in } (t, \theta)$$

for $v \in I_0$. In particular, we have

$$v(t_0 + T_v(\theta), \Xi(\theta)) = \begin{cases} 1 & \text{if } d_4(v) > 0 \text{ or } \hat{a}_3(t, \Xi(\theta)) \equiv 0 \text{ in } (t, \theta), \\ -1 & \text{if } d_4(v) < 0 \end{cases}$$

for $v \in I_0$, where

$$v(t, \xi) = \begin{cases} 1 & \text{if } \hat{a}_3(t, \xi) \geq 0, \\ -1 & \text{if } \hat{a}_3(t, \xi) < 0. \end{cases}$$

We replace $T(\theta)$ by $T_{v_0}(\theta)$. Then we can assume that $I_0 = [-s_0, s_0]$, $\mu_l = \tilde{\mu}_l$ $(1 \leq l \leq 3)$ and $\min\{\mu_1, \mu_3\} < \hat{\mu}$ or $\mu_2 < \hat{\mu}_0$. Let κ and δ' be positive rational constants satisfying $\delta'\kappa < 1$. Moreover, we assume that $\delta' \in (0, 1)$ and $1 - \delta'\kappa < \delta\kappa/2$ (see (3.18) below). We make an asymptotic change of variables:

$$t = t(s; \rho) \equiv t_0 + T(\rho^{-\kappa}) + \rho^{-\delta\kappa} s, \quad x = x(y; \rho) \equiv \rho^{\delta'\kappa - 1} y. \tag{3.6}$$

Put

$$P_\rho(s, \sigma, \eta) = P(t(s; \rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta). \tag{3.7}$$

Let K be a compact neighborhood of $(t_0, 0)$ in $\mathbf{R} \times \mathbf{R}^n$, and put

$$V = \{(s, y, \rho^{-1}) \in [-s_0, s_0] \times \mathbf{R}^n \times (0, \rho_0^{-1}];\ |y| \leq 1\},$$

where $\rho_0 > 0$. We choose ρ_0 so that

$$\begin{aligned} &\{(t(s; \rho), x(y; \rho));\ s \in [-s_0, s_0] \text{ and } |y| \leq 1\} \\ &\qquad \subset \{(t, x) \in K;\ t \in [0, \delta_1]\} \quad \text{for } \rho \geq \rho_0. \end{aligned} \tag{3.8}$$

LEMMA 3.3. *Let $\psi \in C^\infty(\mathbf{R})$, and let $q(s, \sigma)$ be a polynomial of σ of degree* 3. *Then we have*

$$\begin{aligned} &e^{-i\psi(s)} q(s, \rho^{\delta\kappa} D_s)(e^{i\psi(s)} u(s)) \\ &\quad = \Big[q(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma)) - \frac{i}{2} q^{(2)}(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma))\rho^{2\delta\kappa}\partial_s^2\psi(s) \\ &\qquad - \frac{1}{6} q^{(3)}(s, \rho^{\delta\kappa}(\partial_s\psi(s) + \sigma))\rho^{3\delta\kappa}\partial_s^3\psi(s) \Big]_{\sigma = D_s} u(s) \end{aligned}$$

for $u(s) \in C^\infty(\mathbf{R})$, where $q^{(k)}(s, \sigma) = \partial_\sigma^k q(s, \sigma)$. Here $a(s, \sigma)|_{\sigma = D_s} = a(s, D_s)$ for a symbol $a(s, \sigma)$.

Proof. If $q(s,\sigma) = \sigma$, σ^2 or σ^3, then the lemma can be easily proved. This proves the lemma. $\square$

Let $\varepsilon = \pm 1$, and let v_0 and γ_0 be positive constants. Put

$$\varphi(s;\rho) = \sum_{k=0}^{\hat{l}} \rho^{-k\gamma_0} \varphi_k(s;\rho), \tag{3.9}$$

$$\Phi(s,y;\rho) = \rho^{1-\delta\kappa} \int_0^s \{A(t(u;\rho), \Xi(\rho^{-\kappa})) - a_1(t(u;\rho), \Xi(\rho^{-\kappa}))/3\}\, du + \rho^{\delta'\kappa} y \cdot \Xi(\rho^{-\kappa}),$$

$$E(s,y;\rho,\varepsilon,v_0,\varphi) = \exp[i\varepsilon\Phi(s,y;\rho) + i\rho^{v_0}\varphi(s;\rho)], \tag{3.10}$$

where $\varphi_k(s;\rho) \in C^\infty([-s_0,s_0])$ for $\rho \geq \rho_0$, the $\varphi_k(s;\rho)$ satisfy $|\partial_s^l \varphi_k(s;\rho)| \leq C_l$ for $l \in \mathbf{Z}_+$ and $(s,\rho^{-1}) \in [-s_0,s_0] \times (0,\rho_0^{-1}]$, $\hat{l} = 0$ or 1, and $A(t,\xi)$ $(\equiv A^{k_0}(t,\xi))$ is defined by (2.11) with $k = k_0$. By Lemma 3.3 we have

$$\begin{aligned}
&\tilde{P}(s,D_s;\rho,E)u(s) \\
&\quad \equiv E(s,y;\rho,\varepsilon,v_0,\varphi)^{-1} P_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,v_0,\varphi)u(s)) \\
&\quad = E(s,0;\rho,\varepsilon,v_0,\varphi)^{-1} P(t(s;\rho), \rho^{\delta\kappa} D_s, \varepsilon\rho\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,v_0,\varphi)u(s)) \\
&\quad = \Big[P(t(s;\rho), \varepsilon\rho\tilde{A}(s;\rho) + \rho^{\delta\kappa+v_0}\partial_s\varphi + \rho^{\delta\kappa}\sigma, \varepsilon\rho\Xi(\rho^{-\kappa})) \\
&\qquad - \frac{i}{2} P^{(2)}(t(s;\rho), \varepsilon\rho\tilde{A} + \rho^{\delta\kappa+v_0}\partial_s\varphi + \rho^{\delta\kappa}\sigma, \varepsilon\rho\Xi(\rho^{-\kappa})) \\
&\qquad \times \rho^{2\delta\kappa}(\varepsilon\rho^{1-\delta\kappa}\partial_s\tilde{A} + \rho^{v_0}\partial_s^2\varphi) - \rho^{3\delta\kappa}(\varepsilon\rho^{1-\delta\kappa}\partial_s^2\tilde{A} + \rho^{v_0}\partial_s^3\varphi) \Big]_{\sigma=D_s} u(s),
\end{aligned}$$

where

$$\tilde{A} \equiv \tilde{A}(s;\rho) = A(t(s;\rho), \Xi(\rho^{-\kappa})) - a_1(t(s;\rho), \Xi(\rho^{-\kappa}))/3$$

and $\varphi = \varphi(s;\rho)$.

Lemma 3.4. *Let $\mu \in \mathbf{Z}_+$, and let $a(s,\theta) \in C^\infty([-s_0,s_0] \times [0,\theta_0])$ satisfy*

$$a(s,\theta) = O(\theta^\mu) \quad \textit{uniformly in } s \in [-s_0,s_0] \textit{ as } \theta \downarrow 0.$$

Namely, there is $C > 0$ such that

$$|\theta^{-\mu} a(s,\theta)| \leq C \quad \textit{if} \quad (s,\theta) \in [-s_0,s_0] \times (0,\theta_0].$$

Then, for any $l \in \mathbf{Z}_+$

$$\partial_s^l a(s,\theta) = O(\theta^\mu) \quad \textit{uniformly in } s \in [-s_0, s_0] \textit{ as } \theta \downarrow 0.$$

REMARK. For instance, for $\hat{a}_2(t,\xi)$ there is $L \in \mathbf{N}$ such that

$$a(s,\theta) \equiv \hat{a}_2(t(s;\theta^{-L/\kappa}), \Xi(\theta^L)) \in C^\infty([-s_0, s_0] \times [0, \theta_0^{1/L}]).$$

Then, we can apply the lemma to $a(s,\theta)$, and for any $l \in \mathbf{Z}_+$ we have

$$\partial_s^l \hat{a}_2(t(s;\rho), \Xi(\rho^{-\kappa})) = O(\rho^{-\hat{\mu}\kappa}) \quad \text{uniformly in } s \in [-s_0, s_0] \text{ as } \rho \to \infty.$$

PROOF. By assumption we have

$$(\partial_\theta^l a)(s,0) \equiv 0 \quad \text{in } s \ (0 \le l \le \mu - 1).$$

Then Taylor's formula yields

$$\partial_s^l a(s,\theta) = \frac{\theta^\mu}{(\mu-1)!} \int_0^1 (1-\tau)^{\mu-1} (\partial_s^l \partial_\theta^\mu a)(s, \tau\theta)\, d\tau.$$

This proves the lemma. □

Recall that

$$\hat{p}(t,\tau,\xi) = p(t, \tau - a_1(t,\xi)/3, \xi) = \tau^3 - \hat{a}_2(t,\xi)\tau + \hat{a}_3(t,\xi),$$

$$P(t,\tau,\xi) = p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi) + r(t,\tau,\xi),$$

$$q_0(t, \tau - a_1(t,\xi)/3, \xi) = b_0(t,\xi)\tau^2 + \hat{b}_1(t,\xi)\tau + \hat{b}_2(t,\xi),$$

$$\alpha(t,\xi) = \hat{b}_1(t,\xi) + i\partial_t a_1(t,\xi).$$

A straightforward calculation yields

$$\begin{aligned} (3.11) \quad & \tilde{P}(s, D_s; \rho, E)u(s) \\ &= \Big[\varepsilon\rho^3 \hat{p}(t(s,\rho), A(s;\rho), \Xi(\rho^{-\kappa})) + 3\varepsilon\rho^{1+2\delta\kappa} A(s,\rho)(\rho^{\nu_0}\partial_s\varphi + \sigma)^2 \\ &\quad + \rho^{3\delta\kappa}(\rho^{\nu_0}\partial_s\varphi + \sigma)^3 + \rho^2 q_0(t(s,\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa})) \\ &\quad + \varepsilon\rho^{1+\delta\kappa} q_0^{(1)}(t(s,\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi + \sigma) \\ &\quad + \rho^{2\delta\kappa} b_0(t(s,\rho), \Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi + \sigma)^2 \end{aligned}$$

$$+ \varepsilon\rho q_1(t(s,\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa}))$$

$$+ \rho^{\delta\kappa} q_1^{(1)}(t(s,\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi + \sigma)$$

$$+ \frac{\varepsilon}{2}\rho^{2\delta\kappa-1} q_1^{(2)}(t(s,\rho), 0, \Xi(\rho^{-\kappa}))(\rho^{\nu_0}\partial_s\varphi + \sigma)^2$$

$$+ r(t(s;\rho), \varepsilon\rho\tilde{A}(s;\rho) + \rho^{\delta\kappa+\nu_0}\partial_s\varphi + \rho^{\delta\kappa}\sigma, \varepsilon\rho\Xi(\rho^{-\kappa}))$$

$$- \Big\{3\varepsilon i\rho A(s;\rho) + 3i\rho^{\delta\kappa}(\rho^{\nu_0}\partial_s\varphi + \sigma) + ib_0(t(s,\rho), \Xi(\rho^{-\kappa}))$$

$$+ \frac{i}{2} q_1^{(2)}(t(s;\rho), 0, \varepsilon\rho\Xi(\rho^{-\kappa})) + \frac{i}{2} r^{(2)}(t(s;\rho), 0, \varepsilon\rho\Xi(\rho^{-\kappa}))\Big\}$$

$$\times (\varepsilon\rho^{1+\delta\kappa}\partial_s\tilde{A}(s;\rho) + \rho^{2\delta\kappa+\nu_0}\partial_s^2\varphi)$$

$$- (\varepsilon\rho^{1+2\delta\kappa}\partial_s^2\tilde{A}(s;\rho) + \rho^{3\delta\kappa+\nu_0}\partial_s^3\varphi)\Big]_{\sigma=D_s} u(s)$$

$$= [\rho^{3\delta\kappa+3\nu_0}(\partial_s\varphi)^3$$

$$+ \rho^2\{q_0(t(s;\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa})) - 3i\rho^{\delta\kappa} A(s;\rho)\partial_s\tilde{A}(s;\rho)\}$$

$$+ 3\varepsilon\rho^{1+2\delta\kappa+2\nu_0} A(s;\rho)(\partial_s\varphi)^2$$

$$+ \varepsilon\rho^{1+\delta\kappa+\nu_0}\{\hat{b}_1(t(s;\rho), \Xi(\rho^{-\kappa})) + \rho^{\delta\kappa}\partial_s a_1(t(s;\rho), \Xi(\rho^{-\kappa}))\}$$

$$\times (\partial_s\varphi + \rho^{-\nu_0} D_s)$$

$$+ \varepsilon\rho\{q_1(t(s;\rho), \tilde{A}(s;\rho), \Xi(\rho^{-\kappa})) + \rho^{2\delta\kappa}\partial_s^2 a_1(t(s;\rho), \Xi(\rho^{-\kappa}))/3$$

$$+ i\rho^{\delta\kappa} b_0(t(s;\rho), \Xi(\rho^{-\kappa}))\partial_s a_1(t(s;\rho), \Xi(\rho^{-\kappa}))/3\}$$

$$- 2\varepsilon\rho^3(A(s;\rho)^3 - \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2)$$

$$+ \rho^{3\delta\kappa+2\nu_0}\{3(\partial_s\varphi)^2 D_s - 3i(\partial_s\varphi)(\partial_s^2\varphi)$$

$$+ \rho^{-\nu_0} l_1(s, \rho^{-1}; \partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, \varepsilon)$$

$$+ \rho^{-\delta\kappa} b_0(t(s;\rho), \Xi(\rho^{-\kappa}))(\partial_s\varphi)^2 + \rho^{-\delta\kappa-1} l_2(s, \rho^{-1}; \partial_s\varphi, D_s)\}$$

$$+ \varepsilon\rho^{1+2\delta\kappa+\nu_0}\{6A(t(s;\rho), \Xi(\rho^{-\kappa}))(\partial_s\varphi) D_s$$

$$- 3iA(t(s;\rho), \Xi(\rho^{-\kappa}))(\partial_s^2\varphi)$$

$$- 3i\partial_s A(t(s;\rho), \Xi(\rho^{-\kappa}))(\partial_s\varphi) + \rho^{-\hat{\mu}\kappa-\nu_0} l_3(s, \rho^{-1}; D_s)$$

$$+ 2\rho^{-\delta\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa})) b_0(t(s;\rho), \Xi(\rho^{-\kappa}))$$
$$\times (\partial_s\varphi + \rho^{-\nu_0} D_s)\}]u(s)$$
$$= [\rho^{3\delta\kappa+3\nu_0}(\partial_s\varphi)^3 + \rho^{2-\mu_2\kappa+\delta\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(t(s;\rho), \Xi(\rho^{-\kappa})))$$
$$+ 3\varepsilon\rho^{1+2\delta\kappa-\hat{\mu}\kappa+2\nu_0}(\rho^{\hat{\mu}\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi)^2$$
$$+ \varepsilon\rho^{1+2\delta\kappa-\mu_1\kappa+\nu_0}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi + \rho^{-\nu_0} D_s)$$
$$+ \varepsilon\rho^{1-\mu_3\kappa+2\delta\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(s;\rho), \Xi(\rho^{-\kappa})))$$
$$- 2\varepsilon\rho^{3-2\hat{\mu}_0+\hat{\mu}\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(t(s;\rho), \Xi(\rho^{-\kappa}))^3$$
$$- \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2))$$
$$+ \rho^{3\delta\kappa+2\nu_0}\{3(\partial_s\varphi)^2 D_s - 3i(\partial_s\varphi)(\partial_s^2\varphi)$$
$$+ \rho^{-\nu_0} l_1(s, \rho^{-1}; \partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, \varepsilon)$$
$$+ \rho^{-\delta\kappa} b_0(t(s;\rho), \Xi(\rho^{-\kappa}))(\partial_s\varphi)^2 + \rho^{-\delta\kappa-1} l_2(s, \rho^{-1}; \partial_s\varphi, D_s)\}$$
$$+ \varepsilon\rho^{1+2\delta\kappa-\hat{\mu}\kappa+\nu_0}\{6(\rho^{\hat{\mu}\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi) D_s$$
$$- 3i(\rho^{\hat{\mu}\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s^2\varphi)$$
$$- 3i(\rho^{\hat{\mu}\kappa}\partial_s A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi) + \rho^{-\nu_0} l_3(s, \rho^{-1}; D_s)$$
$$+ 2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa} A(t(s;\rho), \Xi(\rho^{-\kappa}))) b_0(t(s;\rho), \Xi(\rho^{-\kappa}))$$
$$\times (\partial_s\varphi + \rho^{-\nu_0} D_s)\}]u(s),$$

where $l_1(s, \theta; X_1, X_2, X_3, X_4, X_5)$ is a polynomial of $\{X_k\}_{1 \le k \le 5}$ with coefficients in $C^\infty([-s_0, s_0] \times [0, 1])$, $\deg_{X_1} l_1 = 2$, $\deg_{X_k} l_1 = 1$ $(k = 2, 3, 5)$ and $\deg_{X_4} l_1 = 3$, $l_2(s, \theta; X_1, X_2)$ is a polynomial of X_1 and X_2 with coefficients in $C^\infty([-s_0, s_0] \times [0, 1])$, $\deg_{X_k} l_2 = 2$ $(k = 1, 2)$, and $l_3(s, \theta; D_s)$ is a differential operator of order 2 with coefficients in $C^\infty([-s_0, s_0] \times [0, 1])$, and $l_3(s, \theta; D_s) = 0$ if $\hat{\mu} = \infty$. Here we have used the facts that

$$3A(t, \xi)^2 = \hat{a}_2(t, \xi),$$

$$\beta(\cdot) = q_0(t(s;\rho), A(\cdot) - a_1(\cdot)/3, \Xi(\rho^{-\kappa})) + i\rho^{\delta\kappa} A(\cdot)\partial_s a_1(\cdot) - \frac{i}{2}\rho^{\delta\kappa}\partial_s \hat{a}_2(\cdot)$$
$$= q_0(t(s;\rho), A(\cdot) - a_1(\cdot)/3, \Xi(\rho^{-\kappa})) - 3i\rho^{\delta\kappa} A(\cdot)\partial_s(A(\cdot) - a_1(\cdot)/3),$$

$$\hat{b}_1(\cdot) + i\rho^{\delta\kappa}\partial_s a_1(\cdot) = \alpha(\cdot),$$

$$q_1(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa})) + \rho^{2\delta\kappa}\partial_s^2 a_1(\cdot)/3 + i\rho^{\delta\kappa} b_0(\cdot)\partial_s a_1(\cdot)/3 = \hat{c}_1(\cdot),$$

$$\begin{aligned} q_1(t(s;\rho), \tilde{A}(\cdot), \Xi(\rho^{-\kappa})) &= q_1(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa})) + q_1^{(1)}(t(s;\rho), -a_1(\cdot)/3, \Xi(\rho^{-\kappa}))A(\cdot) \\ &\quad + \frac{1}{2} q_1^{(2)}(t(s;\rho), 0, \Xi(\rho^{-\kappa}))A(\cdot)^2, \end{aligned}$$

$$\begin{cases} r(t(s;\rho), \varepsilon\rho\tilde{A}(\cdot), \varepsilon\rho\Xi(\rho^{-\kappa})) = O(1), \\ r^{(1)}(t(s;\rho), \varepsilon\rho\tilde{A}(\cdot), \varepsilon\rho\Xi(\rho^{-\kappa})) = O(\rho^{-1}), \\ r^{(2)}(t(s;\rho), 0, \varepsilon\rho\Xi(\rho^{-\kappa})) = O(\rho^{-2}) \end{cases}$$

uniformly in $s \in [-s_0, s_0]$ as $\rho \to \infty$,

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. We note that

$$A(t,\xi)^3 - \hat{a}_3(t,\xi)/2 = v(t,\xi)\{(\hat{a}_2(t,\xi)/3)^{3/2} - |\hat{a}_3(t,\xi)|/2\},$$

$$D_3(t,\xi) = 108\{(\hat{a}_2(t,\xi)/3)^{3/2} - |\hat{a}_3(t,\xi)|/2\}\{(\hat{a}_2(t,\xi)/3)^{3/2} + |\hat{a}_3(t,\xi)|/2\},$$

$$D_3(t,\xi) \le 216|A(t,\xi)^3 - \hat{a}_3(t,\xi)/2|(\hat{a}_2(t,\xi)/3)^{3/2} \le 2D_3(t,\xi).$$

This implies that there is $C > 0$ satisfying

$$\rho^{2\hat{\mu}_0\kappa - \hat{\mu}\kappa}|A(t(s;\rho), \Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2| \le C$$

for $(s, \rho^{-1}) \in [-s_0, s_0] \times (0, \rho_0^{-1}]$. We shall prove Proposition 3.1 by dividing into four cases:

Case A is the case where

$$\min\{\mu_1, \mu_3\} \ge \mu_2/2 \quad \text{and} \quad \mu_2 < 2\hat{\mu}.$$

Case B is the case where

$$\min\{\mu_1, \mu_3\} \ge \hat{\mu} \quad \text{and} \quad 2\hat{\mu} \le \mu_2 < \hat{\mu}_0.$$

Case C is the case where

$$\mu_1 \le \mu_3, \quad 2\mu_1 < \mu_2 \quad \text{and} \quad \mu_1 < \hat{\mu}.$$

Case D is the case where

$$\mu_3 \le \mu_1, \quad 2\mu_3 < \mu_2 \quad \text{and} \quad \mu_3 < \hat{\mu}.$$

Let us first consider Case A. We choose

$$v_0 = (2 - \mu_2\kappa - 2\delta\kappa)/3, \quad \kappa = (\delta + \mu_2/2 + 3\mu_4/2)^{-1},$$

where $\mu_4 = \min\{\hat{\mu} - \mu_2/2, 2/3\}$. Then we have

$$3\delta\kappa + 3\nu_0 = 2 - \mu_2\kappa + \delta\kappa, \quad 1 - \delta\kappa = (\mu_2 + 3\mu_4)\kappa/2,$$

$$\nu_0 = \mu_4\kappa\ (> 0), \quad \nu_0 \le 2/3,$$

$$3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0/2 + (\hat{\mu} - \mu_2/2 - \mu_4)\kappa \ge \nu_0/2,$$

$$3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0/2 + (\mu_1 - \mu_2/2)\kappa \ge \nu_0/2,$$

$$3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_3\kappa) = 3\nu_0/2 + (\mu_3 - \mu_2/2)\kappa \ge 3\nu_0/2,$$

$$3\delta\kappa + 3\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) = 3\nu_0/2 + 2(\hat{\mu}_0 - 2\hat{\mu})\kappa + 3(\hat{\mu} - \mu_2/2 - \mu_4)\kappa \ge 3\nu_0/2.$$

So we choose $\varepsilon = 1$, $\hat{l} = 1$ and $\gamma_0 = \nu_0/2$ in (3.9) and (3.10). We note that

$$A(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv A(t(s;\rho), \Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho), \Xi(\rho^{-\kappa}))/2 \equiv 0$$

and $3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa = -\infty$ when $\hat{\mu} = \infty$. Define $\varphi_0(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi_0(s;\rho) = \int_0^s (-\rho^{\mu_2\kappa - \delta\kappa}\beta(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/3}\, du. \tag{3.12}$$

Note that

$$\rho^{\mu_2\kappa-\delta\kappa}\beta(t(s;\rho), \Xi(\rho^{-\kappa})) = d_2(s) + o(1) \quad \text{as } \rho \to \infty,$$

where $d_2(s) \ne 0$ for $s \in [-s_0, s_0]$. Here we have chozen a branch of $(-\rho^{\mu_2\kappa-\delta\kappa} \times \beta(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/3}$ so that its imaginary part is negative. Then there is $\hat{d} > 0$ such that

$$\operatorname{Im} \varphi_0(s;\rho) \ge \hat{d}|s| \quad \text{for } s \in [-s_0, 0) \text{ and } \rho \ge \rho_0,$$

with a modification of ρ_0 if necessary. Since

$$\begin{aligned}(\partial_s\varphi_0(s;\rho) + \rho^{-\nu_0/2}\partial_s\varphi_1(s;\rho))^3 &= (\partial_s\varphi_0)^3 + 3\rho^{-\nu_0/2}(\partial_s\varphi_0)^2(\partial_s\varphi_1) \\ &\quad + 3\rho^{-\nu_0}(\partial_s\varphi_0)(\partial_s\varphi_1)^2 + \rho^{-3\nu_0/2}(\partial_s\varphi_1)^3,\end{aligned}$$

$\partial_s\varphi_1(s;\rho)$ is chosen so as to satisfy

$$\begin{aligned}&3(\partial_s\varphi_0)^2(\partial_s\varphi_1) + 3\rho^{-(\hat{\mu}-\mu_2/2-\mu_4)}(\rho^{\hat{\mu}\kappa}A(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi_0)^2 \\ &\quad + \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho), \Xi(\rho^{-\kappa})))(\partial_s\varphi_0) = 0.\end{aligned}$$

Noting $\partial_s\varphi_0(s;\rho) = (-d_2(s))^{1/3} + O(1)$ as $\rho \to \infty$, we define

$$\varphi_1(s;\rho) = -\int_0^s [\rho^{-(\hat{\mu}-\mu_2/2-\mu_4)}(\rho^{\hat{\mu}\kappa}A(t(u;\rho),\Xi(\rho^{-\kappa})))$$
$$+ \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho),\Xi(\rho^{-\kappa})))/(3(\partial_s\varphi_0)(u;\rho))]\, du.$$

Putting

$$u(s;\rho^{-1}) \sim \sum_{l=0}^{\infty} \rho^{-l\nu_0} u_l(s;\rho^{-1}), \tag{3.13}$$

$$u_{-1}(s;\rho^{-1}) \equiv 0, \quad u_0(0;\rho^{-1}) = 1, \quad u_k(0;\rho^{-1}) = 0 \ (k \geq 1), \tag{3.14}$$

we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{(3(\partial_s\varphi(s;\rho))^2 + 6\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)\\
&\qquad + \rho^{-\nu_0/2-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))D_s\\
&\quad + 3(\partial_s\varphi_1(s;\rho))^2\partial_s\varphi_0(s;\rho) + \rho^{-\nu_0/2}(\partial_s\varphi_1)^3\\
&\quad + 6\rho^{-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi_0)(\partial_s\varphi_1)\\
&\quad + 3\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi_1)^2\\
&\quad + \rho^{-(\mu_1-\mu_2/2)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi_1)\\
&\quad + \rho^{-\nu_0/2-(\mu_3-\mu_2/2)\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(s;\rho),\Xi(\rho^{-\kappa}))) - 2\rho^{-\nu_0/2-2(\hat{\mu}_0-2\hat{\mu})\kappa-3\nu_1}\\
&\qquad \times (\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(t(s;\rho),\Xi(\rho^{-\kappa}))^3 - \hat{a}_3(t(s;\rho),\Xi(\rho^{-\kappa}))/2))\\
&\quad - 3i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)^2\\
&\quad - 3i\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s^2\varphi)\\
&\quad - 3i\rho^{-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}\partial_sA(t(s;\rho),\Xi(\rho^{-\kappa})))(\partial_s\varphi)\\
&\quad + 2\rho^{-\delta\kappa-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))(\partial_s\varphi)\}\\
&\quad \times u_k(s;\rho^{-1})\\
&+ \{l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,1)\\
&\quad + \rho^{\nu_0-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s) + \rho^{-\nu_0/2-\nu_1}l_3(s,\rho^{-1},D_s)\\
&\quad + 2\rho^{-\delta\kappa-\nu_0/2-\nu_1}(\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa}))b_0(t(s;\rho),\Xi(\rho^{-\kappa}))D_s\}\\
&\quad \times u_{k-1}(s;\rho^{-1}) = 0 \quad (k = 0,1,2,\ldots),
\end{aligned} \tag{3.15}$$

where $\nu_1 = (\hat{\mu} - \mu_2/2 - \mu_4)\kappa$. We can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\dots}$, inductively, so as to satisfy (3.14) and (3.15). It is easy to see that there are $C_{l,k} > 0$ $(l,k \in \mathbf{Z}_+)$ satisfying

$$|D_s^l u_k(s;\rho^{-1})| \le C_{l,k} \quad \text{for } l,k \in \mathbf{Z}_+,\ s \in [-s_0, s_0] \text{ and } \rho \in [\rho_0, \infty).$$

Let $\phi(s) \in C_0^\infty(\mathbf{R})$ satisfy

$$\phi(s) = \begin{cases} 1 & \text{if } |s| \le s_0/2, \\ 0 & \text{if } |s| \ge s_0, \end{cases}$$

and put

$$v_N(s,y;\rho^{-1},\varepsilon) = \sum_{k=0}^{N} \rho^{-k\nu_0} u_k(s;\rho^{-1})\phi(s)E(s,y;\rho,\varepsilon,\nu_0,\varphi) \quad (N \in \mathbf{Z}_+). \tag{3.16}$$

Then we have

$$\begin{aligned}(\rho^{\delta\kappa}D_s)^l(\rho^{1-\delta'\kappa}D_y)^\alpha P_\rho(s,D_s,D_y)v_N(s,y;\rho^{-1},1) \\ = \begin{cases} O(\rho^{3\delta\kappa+2\nu_0-\nu_0(N+1)+l+|\alpha|}) \\ \quad \text{uniformly in } \tilde{\Omega}_{\varepsilon_0,\rho}(0,0) \cap \{|s| \le s_0/2\} \text{ as } \rho \to \infty, \\ O(\rho^{-M}) \\ \quad \text{uniformly in } \tilde{\Omega}_{\varepsilon_0,\rho}(0,0) \cap \{s_0/2 \le |s| \le s_0\} \text{ as } \rho \to \infty \\ \quad (M \in \mathbf{N}), \end{cases}\end{aligned} \tag{3.17}$$

where

$$\tilde{\Omega}_{\varepsilon_0,\rho}(0,0) = \{(s,y) \in \mathbf{R}^{n+1};\ s < -\varepsilon_0\rho^{\delta\kappa-2+2\delta'\kappa}|y|^2\}. \tag{3.18}$$

Here we have taken $\varepsilon = 1$ in Case A. Next cosider Case B. Note that $\hat{\mu} < \infty$ and $\mu_2 < \infty$. We choose

$$\nu_0 = (1 - \delta\kappa + \hat{\mu}\kappa - \mu_2\kappa)/2, \quad \kappa = (\delta - \hat{\mu} + \hat{\mu}_0)^{-1}.$$

Then we have

$$2 - \mu_2\kappa + \delta\kappa = 1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0, \quad 1 - \delta\kappa = (\hat{\mu}_0 - \hat{\mu})\kappa,$$

$$\nu_0 = (\hat{\mu}_0 - \mu_2)\kappa/2\ (> 0),$$

$$1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (3\delta\kappa + 3\nu_0) = \nu_0 + (\mu_2 - 2\hat{\mu})\kappa \ge \nu_0,$$

$$1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0 + (\mu_1 - \hat{\mu})\kappa \ge \nu_0,$$

$$1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (1 - \mu_3\kappa + 2\delta\kappa) = 2\nu_0 + (\mu_3 - \hat{\mu})\kappa \ge 2\nu_0,$$

$$1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) = 2\nu_0.$$

Therefore, we choose $\hat{l} = 0$ in (3.9) and $\varepsilon = \pm 1$ so that the imaginary part of a branch of $(-\varepsilon\tilde{\beta}(s;\rho)/3)^{1/2}$ is negative, where

$$\tilde{\beta}(s;\rho) = (\rho^{\hat{\mu}\kappa}A(t(s;\rho),\Xi(\rho^{-\kappa})))^{-1}(\rho^{\mu_2\kappa-\delta\kappa}\beta(t(s;\rho),\Xi(\rho^{-\kappa}))).$$

We define $\varphi(s;\rho) \in C^\infty([-s_0,s_0]\times[\rho_0,\infty))$ by

$$\varphi(s;\rho) = \int_0^s (-\varepsilon\tilde{\beta}(u;\rho)/3)^{1/2}\,du.$$

Here we have

$$\tilde{\beta}(s;\rho) = (d(s)/3)^{-1/2}d_2(s) + o(1) \quad \text{as } \rho\to\infty,$$

$$d(s) > 0 \quad \text{and} \quad d_2(s) \neq 0 \quad \text{for } s \in [-s_0,s_0],$$

$$\mathrm{Im}(-\varepsilon\tilde{\beta}(s;\rho)/3)^{1/2} < 0 \quad \text{for } s\in[-s_0,s_0].$$

Writing $u(s;\rho^{-1})$ as (3.13), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}(3.19)\qquad &\{6\varepsilon(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)D_s + \rho^{-(\mu_2-2\hat{\mu})\kappa}(\partial_s\varphi)^3\\ &\quad + \varepsilon\rho^{-\kappa(\mu_1-\hat{\mu})}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))(\partial_s\varphi) - 3\varepsilon i(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s^2\varphi)\\ &\quad - 3\varepsilon i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi) + 2\varepsilon\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi)\}u_k(s;\rho^{-1})\\ &+\{\varepsilon\rho^{-\kappa(\mu_1-\hat{\mu})}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))D_s + \varepsilon\rho^{-\kappa(\mu_3-\hat{\mu})}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(\cdot))\\ &\quad - 2\varepsilon(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2))\\ &\quad + 3\rho^{-\kappa(\mu_2-2\hat{\mu})}((\partial_s\varphi)^2D_s - i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2\\ &\qquad + \rho^{-\nu_0}l_1(s,\rho^{-1};\partial_s\varphi,\partial_s^2\varphi,\partial_s^3\varphi,D_s,\varepsilon) + \rho^{-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi,D_s))\\ &\quad + 2\varepsilon\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)D_s + \varepsilon l_3(s,\rho^{-1};D_s)\}u_{k-1}(s;\rho^{-1}) = 0\\ &\quad (k=0,1,2,\ldots),\end{aligned}$$

where $(\cdot) = (t(s;\rho),\Xi(\rho^{-\kappa}))$. Similarly, we can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\ldots}$ so as to satisfy (3.14) and (3.19). We define $v_N(s,y;\rho^{-1},\varepsilon)$ $(N\in\mathbf{Z}_+)$ by (3.16). Then we have (3.17), replacing 1 by ε on the left-hand side and $O(\rho^{3\delta\kappa+2\nu_0-\nu_0(N+1)+l+|\alpha|})$ by $O(\rho^{2-\mu_2\kappa+\delta\kappa-\nu_0(N+2)+l+|\alpha|})$ on the right-hand side. Let us consider Case C. Note that $\mu_1 < \infty$ and $\min\{\mu_2-\mu_1,\hat{\mu}\} > 0$. We choose

$$\nu_0 = (1-\delta\kappa-\mu_1\kappa)/2, \quad \kappa = (\delta+\mu_1+\mu_5)^{-1},$$

where

$$\mu_5 = \min\{\mu_2 - 2\mu_1, \hat{\mu} - \mu_1, 1\}\kappa.$$

Then we have

$$3\delta\kappa + 3\nu_0 = 1 + 2\delta\kappa - \mu_1\kappa + \nu_0, \quad 1 - \delta\kappa = \mu_1\kappa + \mu_5\kappa,$$

$$\nu_0 = (1 - \delta\kappa - \mu_1\kappa)/2 = \mu_5\kappa/2 \ (> 0), \quad \nu_0 \le 1/2,$$

$$3\delta\kappa + 3\nu_0 - (2 + \delta\kappa - \mu_2\kappa) = \nu_0 + (\mu_2 - 2\mu_1 - \mu_5)\kappa \ge \nu_0,$$

$$3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0 + (\hat{\mu} - \mu_1 - \mu_5)\kappa \ge \nu_0,$$

$$3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_3\kappa) = \nu_0 + (\mu_3 - \mu_1)\kappa \ge \nu_0,$$

$$3\delta\kappa + 3\nu_0 - (3 - 2\hat{\mu}_0\kappa + \hat{\mu}\kappa) = 3\nu_0 + 2(\hat{\mu}_0 - 2\hat{\mu})\kappa + 3(\hat{\mu} - \mu_1 - \mu_5)\kappa \ge 3\nu_0.$$

We note that

$$\beta(t(s;\rho), \Xi(\rho^{-\kappa})) \equiv 0 \quad \text{in } s \text{ for } \rho \ge \rho_0 \quad \text{when } \mu_2 = \infty.$$

So we choose $\hat{l} = 0$ in (3.9) and $\varepsilon = \pm 1$ so that the imaginary part of a branch of $(-\varepsilon\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(s;\rho), \Xi(\rho^{-\kappa})))^{1/2}$ is negative. We define $\varphi(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi(s;\rho) = \int_0^s (-\varepsilon\rho^{\mu_1\kappa-\delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))^{1/2}\, du. \tag{3.20}$$

Writing $u(s;\rho^{-1})$ as (3.13), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{2(\partial_s\varphi)^2 D_s + \rho^{-(\mu_2-2\mu_1-\mu_5)\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(\cdot)) \\
&\quad + 3\varepsilon\rho^{-(\hat{\mu}-\mu_1-\mu_5)\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)^2 + \varepsilon\rho^{-(\mu_3-\mu_1)\kappa}(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(\cdot)) \\
&\quad - 3i(\partial_s\varphi)(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2\}u_k(s;\rho^{-1}) \\
&+ \{-2\varepsilon\rho^{-\nu_0-3(\hat{\mu}-\mu_1-\mu_5)\kappa-2(\hat{\mu}_0-2\hat{\mu})\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2)) \\
&\quad + l_1(s,\rho^{-1};\partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, \varepsilon) + \rho^{\nu_0-\delta\kappa-1}l_2(s,\rho^{-1};\partial_s\varphi, D_s) \\
&\quad + \varepsilon\rho^{-(\hat{\mu}-\mu_1-\mu_5)\kappa}(6(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)D_s - 3i(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s^2\varphi) \\
&\qquad - 3i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi) + \rho^{-\nu_0}l_3(s,\rho^{-1};D_s) \\
&\qquad + 2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi + \rho^{-\nu_0}D_s))\} \\
&\quad \times u_{k-1}(s;\rho^{-1}) = 0 \quad (k = 0, 1, 2, \ldots),
\end{aligned} \tag{3.21}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. Similarly, we can determine $\{u_k(s;\rho^{-1})\}_{k=0,1,2,\dots}$ so as to satisfy (3.14) and (3.21). We define $v_N(s,y;\rho^{-1},\varepsilon)$ $(N \in \mathbf{Z}_+)$ by (3.16). Then we have (3.17) with an obvious modification. Let us finally consider Case D. Note that $\mu_3 < \infty$. We choose

$$\nu_0 = (1 - \delta\kappa - \mu_3\kappa)/3, \quad \kappa = (\delta + \mu_3 + \mu_6)^{-1},$$

where

$$\mu_6 = \min\{\mu_2/2 - \mu_3, 3(\hat{\mu} - \mu_3)/4, 3(\mu_1 - \mu_3)/2, 1\}.$$

Then we have

$$\begin{aligned}
&3\delta\kappa + 3\nu_0 = 1 + 2\delta\kappa - \mu_3\kappa,\\
&1 - \delta\kappa = (\mu_3 + \mu_6)\kappa = \mu_3\kappa + 3\nu_0,\\
&\nu_0 = \mu_6\kappa/3\ (> 0), \quad \nu_0 \le 1/3,\\
&3\delta\kappa + 3\nu_0 - (2 - \mu_2\kappa + \delta\kappa) = 3\nu_0 + 2(\mu_2/2 - \mu_3 - \mu_6)\kappa \ge 3\nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \hat{\mu}\kappa + 2\nu_0) = \nu_0 + (\hat{\mu} - \mu_3 - \mu_6)\kappa > \nu_0,\\
&3\delta\kappa + 3\nu_0 - (1 + 2\delta\kappa - \mu_1\kappa + \nu_0) = \nu_0 + (\mu_1 - \mu_3 - 2\mu_6/3)\kappa \ge \nu_0,\\
&3\delta\kappa + 3\nu_0 - (3 + \hat{\mu}\kappa - 2\hat{\mu}_0\kappa) = 6\nu_0 + (2(\hat{\mu}_0 - 2\hat{\mu}) + 3(\hat{\mu} - \mu_3) - 4\mu_6)\kappa \ge 6\nu_0.
\end{aligned}$$

We choose $\varepsilon = 1$ and $\hat{l} = 0$ in (3.9) and (3.10). Define $\varphi(s;\rho) \in C^\infty([-s_0, s_0] \times [\rho_0, \infty))$ by

$$\varphi(s;\rho) = \int_0^s [-(\rho^{\mu_3\kappa - 2\delta\kappa}\hat{c}_1(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/3}\, du. \tag{3.22}$$

Here we have chosen a branch of $[-(\rho^{\mu_3\kappa-2\delta\kappa}\hat{c}_1(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/3}$ so that its imaginary part is negative. Writing $u(s;\rho^{-1})$ as (3.13), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{3(\partial_s\varphi)^2 D_s - 3i(\partial_s\varphi)(\partial_s^2\varphi) + 3\rho^{-(\hat{\mu}-\mu_3-\mu_6)\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))(\partial_s\varphi)^2\\
&\qquad + \rho^{-(\mu_1-\mu_3-2\mu_6/3)\kappa}(\rho^{\mu_1\kappa-\delta\kappa}\alpha(\cdot))(\partial_s\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)^2\}u_k(s;\rho^{-1})\\
&\quad + \{\rho^{-(\hat{\mu}-\mu_3-\mu_6)\kappa}(6(\rho^{\hat{\mu}\kappa}A(\cdot)(\partial_s\varphi)D_s - 3i(\rho^{\hat{\mu}\kappa}A(\cdot)(\partial_s^2\varphi) - 3i(\rho^{\hat{\mu}\kappa}\partial_s A(\cdot))(\partial_s\varphi)\\
&\qquad + 2\rho^{-\delta\kappa}(\rho^{\hat{\mu}\kappa}A(\cdot))b_0(\cdot)(\partial_s\varphi + \rho^{-\nu_0}D_s) + \rho^{-\nu_0}l_3(s,\rho^{-1};D_s))\\
&\qquad + \rho^{-\nu_0-(\mu_2-2\mu_3-2\mu_6)\kappa}(\rho^{\mu_2\kappa-\delta\kappa}\beta(\cdot))\\
&\qquad - 2\rho^{-4\nu_0-(2(\hat{\mu}_0-2\hat{\mu})+3(\hat{\mu}-\mu_3)-4\mu_6)\kappa}(\rho^{2\hat{\mu}_0\kappa-\hat{\mu}\kappa}(A(\cdot)^3 - \hat{a}_3(\cdot)/2))
\end{aligned}$$

$$+ l_1(s, \rho^{-1}; \partial_s\varphi, \partial_s^2\varphi, \partial_s^3\varphi, D_s, 1)$$
$$+ \rho^{\nu_0-\delta\kappa-1} l_2(s, \rho^{-1}; \partial_s\varphi, D_s)\} u_{k-1}(s; \rho^{-1}) = 0 \quad (k = 0, 1, 2, \ldots).$$

Here $l_3(s; \theta, D_s) = 0$ if $\hat{\mu} = \infty$, $\alpha(t(s; \rho), \Xi(\rho^{-\kappa})) \equiv 0$ in s for $\rho \geq \rho_0$ if $\mu_1 = \infty$, and $\beta(t(s; \rho), \Xi(\rho^{-\kappa})) \equiv 0$ in s for $\rho \geq \rho_0$ if $\mu_2 = \infty$. Similarly, we can construct $\{v_N(s, y; \rho^{-1}, 1)\}_{N \in \mathbf{Z}_+}$ satisfying (3.16) with $\varepsilon = 1$ and (3.17).

The condition (3.4) in Proposition 3.1 is satisfied if and only if at least one of Case A to Case D occur. Indeed, first assume that $\min\{\mu_1, \mu_3\} < \hat{\mu}$. If $\mu_1 \leq \mu_3$ and $2\mu_1 < \mu_2$, then $\mu_1 < \hat{\mu}$ and Case C occurs. If $\mu_1 \leq \mu_3$ and $\mu_2 \leq 2\mu_1$, then $\mu_1 < \hat{\mu}$ and Case A occurs. If $\mu_3 < \mu_1$ and $2\mu_3 < \mu_2$, then $\mu_3 < \hat{\mu}$ and Case D occurs. If $\mu_3 < \mu_1$ and $\mu_2 \leq 2\mu_3$, then $\mu_3 < \hat{\mu}$ and Case A occurs. Next assume that $\min\{\mu_1, \mu_3\} \geq \hat{\mu}$ and $\mu_2 < \hat{\mu}_0$. If $\mu_2 < 2\hat{\mu}$ $(\leq \hat{\mu}_0)$, then Case A occurs. If $2\hat{\mu} \leq \mu_2$ $(< \hat{\mu}_0)$, then Case B occurs. This proves the "only if" part. The converse is obvious.

Now we won't omit "k_0", *i.e.*, $P(t, D_t, D_x)$ denotes the differential operator in §1.

LEMMA 3.5. *Assume that the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* K *be a compact neighborhood of* $(t_0, 0)$ *in* $\mathbf{R} \times \mathbf{R}^n$, *and let* ρ_0 *be a positive constant satisfying* (3.8). *Then for any* $p \in \mathbf{Z}_+$ *there are* $C > 0$ *and* $q \in \mathbf{Z}_+$ *such that*

$$|v|_{p, \tilde{\Omega}_{\varepsilon_0, \rho}(0,0)} \leq C\rho^{q\delta\kappa} |P_\rho(s, D_s, D_y)v|_{q, \tilde{\Omega}_{\varepsilon_0, \rho}(0,0)} \quad \textit{for } \rho \geq \rho_0 \textit{ and} \tag{3.23}$$
$$v(s, y) \in C^\infty(\mathbf{R}^{n+1}) \textit{ with } \operatorname{supp} v \subset \{(s, y);\ t(s; \rho) \geq 0\},$$

where $\tilde{\Omega}_{\varepsilon_0, \rho}(0, 0)$ *is defined by* (3.18), ε_0 *is a positive constant defined as* Lemma 2.5, *and* $P_\rho(s, \sigma, \eta)$ *is defined by* (3.7).

PROOF. Let $v(s, y) \in C^\infty(\mathbf{R}^{n+1})$ satisfy $\operatorname{supp} v \subset W$, where $W = [-s_0, s_0] \times \{y \in \mathbf{R}^n;\ |y| \leq 1\}$. Put

$$u_\rho(t, x) = v(\rho^{\delta\kappa}(t - t_0 - T(\rho^{-\kappa})), \rho^{-\delta'\kappa+1}x).$$

Then we have

$$P(t, D_t, D_x)u_\rho(t, x)|_{t=t(s;\rho),\, x=x(y;\rho)} = P_\rho(s, D_s, D_y)v(s, y).$$

It is obvious that

$$(s, y) \in \tilde{\Omega}_{\varepsilon_0, \rho}(0, 0) \Leftrightarrow (t(s; \rho), x(y; \rho)) \in \Omega_{\varepsilon_0}(t_0 + T(\rho^{-\kappa}), 0).$$

Therefore, Lemma 2.5 proves the lemma. □

We factorized $P(t,\tau,\xi)$ as (3.1). Then we have

$$P_\rho(s,D_s,D_y)(\exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]u(s))$$
$$= \exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]\{P_\rho^1(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))\cdots P_\rho^{k_0-1}(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))$$
$$\times P_\rho^{k_0+1}(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))\cdots P_\rho^{r}(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))$$
$$\times P_\rho^{k_0}(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))+R_\rho(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))\}u(s),$$

where $R_\rho(s,\sigma,\eta)=R(t(s;\rho),\rho^{\delta\kappa}\sigma,\rho^{1-\delta'\kappa}\eta)$. For $P_\rho^{k_0}(s,D_s,D_y)$ we constructed asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N\in\mathbf{Z}_+}$ satisfying (3.16) and (3.17) with an obvious modification when at least one of Case A, Case C and Case D occurs. Here we should choose ε appropriately. In Case B we constructed asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N\in\mathbf{Z}_+}$ satisfying (3.16) and (3.17) with $3\delta\kappa+2\nu_0$ in the exponent replaced by $2-\mu_2\kappa+\delta\kappa$. In (3.17) we replace 1 by ε. Note that

$$P_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,\nu_0,\varphi)u(s))$$
$$=\exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]P_\rho(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,\nu_0,\varphi)u(s)),$$
$$R_\rho(s,D_s,D_y)(E(s,y;\rho,\varepsilon,\nu_0,\varphi)u(s))$$
$$=\exp[i\varepsilon\rho^{\delta'\kappa}y\cdot\Xi(\rho^{-\kappa})]R_\rho(s,D_s,\varepsilon\rho^{\delta'\kappa}\Xi(\rho^{-\kappa}))(E(s,0;\rho,\varepsilon,\nu_0,\varphi)u(s))$$
$$=O(\rho^{-M})\quad\text{uniformly in } [-s_0,0] \text{ as } \rho\to\infty\quad(M\in\mathbf{N})$$

for $u(s)\in C^\infty([-s_0,s_0])$. Therefore, Lemma 3.5 proves Proposition 3.1, since the asymptotic solutions $\{v_N(s,y;\rho^{-1},\varepsilon)\}_{N\in\mathbf{Z}_+}$ violate (3.23).

Lemma 3.6. *Assume that* $1\le k_0\le r$ *and* $m(k_0)=3$, *and that the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* $(t_0,\xi^0)\in[0,\delta_1/2]\times(\Gamma\cap S^{n-1})$, *and let* $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T,Ξ).

(i) *We have*

$$\text{(3.24)}\qquad \mathrm{Ord}_{\theta\downarrow 0}\min_{s\in\mathscr{R}_0(\Xi(\theta);p^{k_0})}|t_0+T(\theta)-s|$$
$$\times\, sub\,\sigma(P)(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))$$
$$\ge \mathrm{Ord}_{\theta\downarrow 0}\,h_{m-1}(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))^{1/2},$$

$$\text{(3.25)}\qquad \mathrm{Ord}_{\theta\downarrow 0}\min_{s\in\mathscr{R}_0(\Xi(\theta);p^{k_0})}|t_0+T(\theta)-s|$$
$$\times\,(\partial_\tau\, sub\,\sigma(P))(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\Xi(\theta))$$

$$\geq \mathrm{Ord}_{\theta\downarrow 0}\, \hat{a}_2^{k_0}(\cdot)^{1/2}$$

$$(= \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-2}(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta))^{1/2}),$$

where $A^{k_0}(t,\xi)$ *is defined by* (2.11) *with* $k = k_0$ *and* $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$.

(ii) *Assume that* $\tau_0 \in \mathbf{R}$ *and* $(\partial_\tau^l p^{k_0})(t_0, \tau_0, \xi^0) = 0$ $(l = 0, 1, 2)$, *and put* $z^0 = (t_0, \tau_0, \xi^0)$. *Then we have*

$$(3.26) \quad \mathrm{Ord}_{\theta\downarrow 0} \min_{s \in \mathscr{R}_0(\Xi(\theta); p^{k_0})} |t_0 + T(\theta) - s|^2 \times Q(t_0 + T(\theta), -a_1(\cdot; z^0)/3, \Xi(\theta); z^0)$$

$$\geq \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-2}(t_0 + T(\theta), -a_1(\cdot; z^0)/3, \Xi(\theta))^{1/2},$$

where $(\cdot; z^0) = (t_0 + T(\theta), \Xi(\theta); z^0)$.

REMARK. (i) On the assumption that the factorization (2.1) is given near $t = 0$, the lemma is stated. Therefore, if for $t_0 \in [0, \infty)$ the factorization of $p(t, \tau, \xi)$ is given in a neighborhood I of t_0, the lemma is valid with $[0, \delta_1/2]$ replaced by a compact sub-interval of $\mathring{I}$. (ii) We note that $p(t, \tau, \xi; z^0) = p^{k_0}(t, \tau, \xi)$ and $a_1(t, \xi; z^0) = a_1^{k_0}(t, \xi)$ in the assertion (ii).

PROOF. From (2.5) of [11] it follows that

$$(3.27) \quad sub\ \sigma(P)(\cdot) = sub\ \sigma(P^{k_0})(\cdot)\Pi_{\{k_0\}}(\cdot) + \sum_{1 \leq k \leq r, k \neq k_0} sub\ \sigma(P^k)(\cdot) p^{k_0}(\cdot)\Pi_{\{k_0, k\}}(\cdot) + O(h_{m-1}(\cdot)^{1/2}),$$

where $(\cdot) = (t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta))$. On the other hand, by (1.1) we have

$$(3.28) \quad h_{m-1}(t, \tau, \xi) = h_2(t, \tau, \xi; p^{k_0})\Pi_{\{k_0\}}(t, \tau, \xi)^2 + h_{m-4}(t, \tau, \xi; p/p^{k_0}) p^{k_0}(t, \tau, \xi)^2,$$

$$(3.29) \quad \mathrm{Ord}_{\theta\downarrow 0}\, p^k(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta)) = 0 \quad \text{if } k \neq k_0,$$

where $h_l(t, \tau, \xi; p) = 0$ if $l < 0$, and $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. Corollary 3.2, (3.27) and (3.28) prove (3.24), since

$$\mathrm{Ord}_{\theta\downarrow 0}\, p^{k_0}(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta))^2$$

$$> \mathrm{Ord}_{\theta\downarrow 0}\, h_2(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta); p^{k_0})$$

if $0 < \mathrm{Ord}_{\theta\downarrow 0}\, p^{k_0}(t_0 + T(\theta), A^{k_0}(\cdot) - a_1^{k_0}(\cdot)/3, \Xi(\theta)) < \infty$, where $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. It follows from (2.5) of [11] that

$$
\begin{aligned}
(3.30)\quad & \partial_\tau\, sub\, \sigma(P)(t,\tau,\xi) \\
&= \partial_\tau\, sub\, \sigma(P^{k_0})(t,\tau,\xi)\cdot \Pi_{\{k_0\}}(t,\tau,\xi) \\
&\quad + sub\, \sigma(P^{k_0})(t,\tau,\xi)\partial_\tau \Pi_{\{k_0\}}(t,\tau,\xi) \\
&\quad + \sum_{1\le k\le r, k\ne k_0} \{\partial_\tau p^{k_0}(t,\tau,\xi)\cdot sub\, \sigma(P^k)(t,\tau,\xi)\Pi_{\{k_0,k\}}(t,\tau,\xi) \\
&\qquad\qquad + p^{k_0}(t,\tau,\xi)\partial_\tau(sub\, \sigma(t,\tau,\xi)\Pi_{\{k_0,k\}}(t,\tau,\xi))\} \\
&\quad - \frac{i}{2}\sum_{1\le k\le r, k\ne k_0} \{\partial_\tau\{p^k, p^{k_0}\}(t,\tau,\xi)\cdot \Pi_{\{k_0,k\}}(t,\tau,\xi) \\
&\qquad\qquad + \{p^k, p^{k_0}\}(t,\tau,\xi)\partial_\tau \Pi_{\{k_0,k\}}(t,\tau,\xi)\} \\
&\quad - \frac{i}{2}\sum_{1\le k<l\le r, k,l\ne k_0} \{\partial_\tau p^{k_0}(t,\tau,\xi)\cdot \{p^k, p^l\}(t,\tau,\xi)\Pi_{\{k_0,k,l\}}(t,\tau,\xi) \\
&\qquad\qquad + p^{k_0}(t,\tau,\xi)\partial_\tau(\{p^k, p^l\}(t,\tau,\xi)\Pi_{\{k_0,k,l\}}(t,\tau,\xi))\}.
\end{aligned}
$$

Corollary 3.2, (3.29) and (3.30) prove (3.25), since

$$
\begin{aligned}
&|(\partial_t^\mu \partial_\tau^\nu p^{k_0})(t,\tau,\xi)| \le C h_{3-l}(t,\tau,\xi; p^{k_0})^{1/2} \quad \text{if } l = 1,2 \text{ and } \mu+\nu = l, \\
&h_{m-2}(t,\tau,\xi) = h_1(t,\tau,\xi; p^{k_0})\Pi_{\{k_0\}}(t,\tau,\xi)^2 + h_2(t,\tau,\xi; p^{k_0})h_{m-4}(t,\tau,\xi; p/p^{k_0}) \\
&\qquad\qquad + h_{m-5}(t,\tau,\xi; p/p^{k_0})p^{k_0}(t,\tau,\xi)^2.
\end{aligned}
$$

Next let us prove the assertion (ii). Corollary 3.2 yields

$$
\mathrm{Ord}_{\theta\downarrow 0} \min_{s\in \mathscr{R}_0(\Xi(\theta); p^{k_0})} |t_0 + T(\theta) - s|^2\, sub^2\, \sigma(P^{k_0})(\cdot) \ge \mathrm{Ord}_{\theta\downarrow 0}\, h_1(\cdot; p^{k_0})^{1/2},
$$

where $(\cdot) = (t_0 + T(\theta), -a_1^{k_0}(t_0 + T(\theta), \Xi(\theta))/3, \Xi(\theta))$. The repetition of the above argument and (2.26) of [11] prove the assertion (ii). □

4. The Double Characteristic Factors

Fix j and k_0 so that $1 \le j \le N_0$, $1 \le k_0 \le r(j)$ and $m(j,k_0) = 2$. In this section we omit the subscript j and the superscript j in the same manner as in §2. We also omit the superscript k_0 until Lemma 4.3. Write

$$p(t,\tau,\xi)\ (=p^{j,k_0}(t,\tau,\xi)) = \tau^2 + a_1(t,\xi)\tau + a_2(t,\xi),$$

$$\hat{p}(t,\tau,\xi) = p(t,\tau - a_1(t,\xi)/2,\xi) = \tau^2 - \hat{a}_2(t,\xi),$$

$$P(t,\tau,\xi) = p(t,\tau,\xi) + q_0(t,\tau,\xi) + q_1(t,\tau,\xi),$$

where $q_0(t,\tau,\xi)$ is positively homogeneous of degree 1 in (τ,ξ) for $|\xi| \geq 1$ and

$$\hat{a}_2(t,\xi) = a_1(t,\xi)^2/4 - a_2(t,\xi)\ (\geq 0),$$

$$q_0(t,\tau,\xi) \equiv b_0(t,\xi)\tau + b_1(t,\xi) \in \mathscr{S}^{1,0}_{1,0}([0,\delta_1] \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\})),$$

$$q_1(t,\tau,\xi) \equiv c_0(t,\xi)\tau + c_1(t,\xi) \in \mathscr{S}^{1,-1}_{cl}([0,\delta_1] \times ((\bar{\Gamma} \cup (-\bar{\Gamma}))\backslash\{0\})).$$

Here we assume that $P(t,\tau,\xi)\ (= P^{k_0}(t,\tau,\xi))$ is defined for $\xi \in (-\bar{\Gamma})\backslash\{0\}$ as stated in §2. We also write

$$\hat{q}_0(t,\tau,\xi) = q_0(t,\tau - a_1(t,\xi)/2,\xi) = b_0(t,\xi)\tau + \hat{b}_1(t,\xi),$$

$$\hat{b}_1(t,\xi) = b_1(t,\xi) - a_1(t,\xi)b_0(t,\xi)/2.$$

Note that

$$h_1(t,\tau - a_1(t,\xi)/2,\xi;p) = h_1(t,\tau,\xi;\hat{p}) = 2\tau^2 + 2\hat{a}_2(t,\xi).$$

Let $t_0 \in [0,\delta_1/2]$, $\xi^0 \in \Gamma \cap S^{n-1}$ and $\theta_0 > 0$, and let $T(\theta), \Xi_l(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ $(1 \leq l \leq n)$ be real-valued functions satisfying the condition (T,Ξ).

(I) The case where $\hat{a}_2(t,\Xi(\theta)) \not\equiv 0$ in (t,θ).

Applying the Weierstrass preparation theorem, we can write

$$\hat{a}_2(t_0 + t,\Xi(\theta)) = \theta^{l_0} d(t,\theta) \prod_{i=1}^{n_0} (t - t_i(\theta)), \quad d(t,\theta) \neq 0$$

for $(t,\theta) \in [-\delta_0,\delta_0] \times [0,\theta_0]$, where $0 < \delta_0 \leq \delta_1 - t_0$ and $t_i(\theta) \equiv t_i(\theta;t_0,\Xi)$. The $t_i(\theta)$ can be expanded into convergent Puiseux series of θ in $[0,\theta_0]$, with a modification of θ_0 if necessary. Note that

$$\mathscr{R}_0(\Xi(\theta)) \supset \{(t_0 + \mathrm{Re}\ t_i(\theta))_+;\ 1 \leq i \leq n_0\}\ (\equiv \mathscr{R}_0(\Xi(\theta);p)) \quad (\theta \in (0,\theta_0]).$$

(II) The case where $\hat{a}_2(t,\Xi(\theta)) \equiv 0$ in (t,θ).

We have $\hat{p}(t,\tau,\Xi(\theta)) = \tau^2$, and put

$$\mathscr{R}_0(\Xi(\theta);p) = \varnothing\ (\subset \mathscr{R}_0(\Xi(\theta))), \quad n_0 = 0 \quad \text{and} \quad l_0 = \infty.$$

Now we define

$$\hat{\mu}\ (\equiv \hat{\mu}(t_0, \xi^0, T, \Xi)) = (\mathrm{Ord}_{\theta\downarrow 0}\ \hat{a}_2(t_0 + T(\theta), \Xi(\theta)))/2,$$

$$\mu_1\ (\equiv \mu_1(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta\downarrow 0} \min_{s\in\mathscr{R}_0(\Xi(\theta);p)} |t_0 + T(\theta) - s|\alpha(t_0 + T(\theta), \Xi(\theta)),$$

$$\delta\ (\equiv \delta(t_0, \xi^0, T, \Xi)) = \mathrm{Ord}_{\theta\downarrow 0} \min_{s\in\mathscr{R}_0(\Xi(\theta);p)} |t_0 + T(\theta) - s|$$

$$\left(= \max_{1\le i\le n_0} \mathrm{Ord}_{\theta\downarrow 0}|t_0 + T(\theta) - (t_0 + \mathrm{Re}\ t_i(\theta))_+|\right),$$

where

$$\alpha(t, \xi) = \hat{b}_1(t, \xi) + i\partial_t a_1(t, \xi)/2$$

and $\delta = 0$ in the case (II).

Proposition 4.1. *If*

$$\mu_1 < \hat{\mu}, \tag{4.1}$$

then the Cauchy problem (CP) *is not* C^∞ *well-posed or* (CP) *does not have finite propagation property.*

Corollary 4.2. *Assume that the Cauchy problem* (CP) *is* C^∞ *well-posed and has finite propagation property. Let* $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma \cap S^{n-1})$. *Then we have*

$$\hat{\mu}(t_0, \xi^0, T, \Xi) \le \mu_1(t_0, \xi^0, T, \Xi)$$

if $T(\theta)$ *and* $\Xi(\theta)$ *satisfy the condition* (T, Ξ).

In the rest of this section we shall prove Proposition 4.1. Assume that (4.1) is satisfied. Then we have $\delta \le \mu_1 < \infty$ and $0 < \hat{\mu}\ (\le \infty)$. There is $c_0 > 0$ such that

$$\min_{1\le i\le n_0} |t_0 + T(\theta) - (t_0 + \mathrm{Re}\ t_i(\theta))_+| \ge c_0\theta^\delta \quad \text{for } \theta \in [0, \theta_0].$$

In the case (II) we may take $c_0 = 1$ since $n_0 = 0$. For $v \in \mathbf{R}$ we define $T_v(\theta)$ by (3.5). In the case (I) we have

$$\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{2\hat{\mu}}(d(v) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $d(v) \not\equiv 0$ is a polynomial of v with real coefficients. It is obvious that

$$d(v) > 0 \quad \text{for } v \in [-c_0/2, c_0/2],$$

$$\hat{a}_2(t_0 + T_v(\theta), \Xi(\theta))^{1/2} = \theta^{\hat{\mu}}(\sqrt{d(v)} + o(1))$$
$$\text{uniformly in } v \in [-c_0/2, c_0/2] \text{ as } \theta \downarrow 0.$$

Noting that $\alpha(t, \Xi(\theta)) \not\equiv 0$ in (t, θ), we write

$$\alpha(t_0 + T_v(\theta), \Xi(\theta)) = \theta^{\tilde{\mu}_1 - \delta}(d_1(v) + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $\tilde{\mu}_1 \in \mathbf{Q}$ and $d_1(v)$ ($\not\equiv 0$) is a polynomial of v. There are $v_0 \in (c_0/4, c_0/2)$ and $s_0 > 0$ such that $I_0 \equiv [v_0 - s_0, v_0 + s_0] \subset [c_0/4, c_0/2]$ and

$$d_1(v) \neq 0 \quad \text{for } v \in I_0.$$

We replace $T(\theta)$ by $T_{v_0}(\theta)$. We note that $\delta = \tilde{\mu}_1 = 0$ and $c_0 = 1$ if $\hat{\mu} = \infty$. Then we can assume that $I_0 = [-s_0, s_0]$, $\mu_1 = \tilde{\mu}_1$ and $\mu_1 < \hat{\mu}$. Similarly, we make an asymptotic change of variables as (3.6), where $\delta' \in (0, 1)$, $\kappa > 0$ and $\delta'\kappa < 1$. Let K be a compact neighborhood of $(t_0, 0)$ in $\mathbf{R} \times \mathbf{R}^n$, and choose $\rho_0 > 0$ so that (3.8) is satisfied. Define

$$P_\rho(s, \sigma, \eta) = P(t(s; \rho), \rho^{\delta\kappa}\sigma, \rho^{1-\delta'\kappa}\eta),$$

and put

$$\Phi(s, y; \rho) = -\rho^{1-\delta\kappa} \int_0^s a_1(t(u; \rho), \Xi(\rho^{-\kappa}))\, du/2 + \rho^{\delta'\kappa} y \cdot \Xi(\rho^{-\kappa}),$$

$$E(s, y; \rho, \varepsilon, v_0, \varphi) = \exp[i\varepsilon\Phi(s, y; \rho) + i\rho^{v_0}\varphi(s; \rho)],$$

where $\varphi(s; \rho)$ ($\in C^\infty([-s_0, s_0])$ for $\rho \geq \rho_0$) satisfies

$$|\partial_s^l \varphi(s; \rho)| \leq C_l \quad \text{for } l \in \mathbf{Z}_+ \text{ and } (s; \rho^{-1}) \in [-s_0, s_0] \times (0, \rho_0^{-1}],$$

$\varepsilon = \pm 1$ and $v_0 > 0$. Applying the same argument as in (3.11), we have

$$\begin{aligned}
\tilde{P}(s, D_s; \rho, E)u(s) &\equiv E(s, y; \rho, v_0, \varphi)^{-1} P_\rho(s, D_s, D_y)(E(s, y; \rho, v_0, \varphi)u(s)) \\
&= E(s, 0; \rho, v_0, \varphi)^{-1} P(t(s; \rho), \rho^{\delta\kappa} D_s, \varepsilon\rho\Xi(\rho^{-\kappa}))(E(s, 0; \rho, v_0, \varphi)u(s)) \\
&= [\rho^{2\delta\kappa + 2v_0}(\partial_s\varphi(s; \rho))^2 + \varepsilon\rho^{1-\mu_1\kappa+\delta\kappa}(\rho^{\mu_1-\delta\kappa}\alpha(\cdot)) \\
&\quad + 2\rho^{2\delta\kappa+v_0}(\partial_s\varphi)D_s + \rho^{2\delta\kappa}D_s^2 - \rho^{2-2\hat{\mu}\kappa}(\rho^{2\hat{\mu}\kappa}\hat{a}_2(\cdot)) \\
&\quad - i\rho^{2\delta\kappa+v_0}(\partial_s^2\varphi) + \rho^{\delta\kappa+v_0} b_0(\cdot)(\partial_s\varphi + \rho^{-v_0}D_s) \\
&\quad + c_0(t(s; \rho), \varepsilon\rho\Xi(\rho^{-\kappa})))(-\varepsilon\rho a_1(\cdot)/2 + \rho^{\delta\kappa+v_0}\partial_s\varphi + \rho^{\delta\kappa}D_s) \\
&\quad + c_1(t(s; \rho), \varepsilon\rho\Xi(\rho^{-\kappa}))]u(s),
\end{aligned}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$. We choose

$$\nu_0 = (1 - \mu_1\kappa - \delta\kappa)/2, \quad \kappa = (\delta + \min\{\mu_1 + 1, \hat{\mu}\})^{-1}.$$

Then we have

$$2\delta\kappa + 2\nu_0 = 1 - \mu_1\kappa + \delta\kappa, \quad 1 - \delta\kappa = \min\{\mu_1 + 1, \hat{\mu}\}\kappa,$$
$$\nu_0 = \min\{1, \hat{\mu} - \mu_1\}\kappa/2 \ (> 0), \quad \nu_0 \le 1/2,$$
$$2\delta\kappa + 2\nu_0 - (2 - 2\hat{\mu}\kappa) = 2\nu_0 - 2\min\{1 + \mu_1 - \hat{\mu}, 0\}\kappa \ge 2\nu_0.$$

Put

$$\varphi(s;\rho) = \int_0^s [-\varepsilon(\rho^{\mu_1\kappa - \delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/2}\, du.$$

Here we have chosen $\varphi(s;\rho)$ and $\varepsilon = \pm 1$ so that

$$\mathrm{Im}[-\varepsilon(\rho^{\mu_1\kappa - \delta\kappa}\alpha(t(u;\rho), \Xi(\rho^{-\kappa})))]^{1/2} < 0.$$

Writing $u(s;\rho^{-1})$ as (3.13), we obtain the following transport equations for $u(s;\rho^{-1})$:

$$\begin{aligned}
&\{2(\partial_s\varphi(s;\rho))D_s - i(\partial_s^2\varphi) + \rho^{-\delta\kappa}b_0(\cdot)(\partial_s\varphi)\}u_k(s;\rho^{-1}) \\
&\quad + \{D_s^2 - \rho^{2\min\{1+\mu_1-\hat{\mu},0\}}(\rho^{2\hat{\mu}\kappa}\hat{a}_2(\cdot)) + \rho^{-\delta\kappa}b_0(\cdot)D_s \\
&\qquad + \rho^{-2\delta\kappa}c_0(\cdots)(-\varepsilon\rho a_1(\cdot)/2 + \rho^{\delta\kappa+\nu_0}\partial_s\varphi + \rho^{\delta\kappa}D_s) \\
&\qquad + \rho^{-2\delta\kappa}c_1(\cdots)\}u_{k-1}(s;\rho^{-1}) \quad (k = 0, 1, 2, \ldots),
\end{aligned}$$

where $(\cdot) = (t(s;\rho), \Xi(\rho^{-\kappa}))$ and $(\cdots) = (t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa}))$. Note that, with some $C_\mu > 0$,

$$|(\rho^{\delta\kappa}\partial_s)^\mu c_l(t(s;\rho), \varepsilon\rho\Xi(\rho^{-\kappa}))| \le C_\mu\rho^{l-1} \quad (l = 0, 1).$$

Therefore, applying the same argument as in §3 we can prove Proposition 4.1. The same argument as in the proof of Lemma 3.6 and Proposition 4.1 prove the following

Lemma 4.3. *Assume that $1 \le k_0 \le r$ and $m(k_0) = 2$, and that* (CP) *is C^∞ well-posed and finite propagation property. Let $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma \cap S^{n-1})$, and let $T(\theta)$ and $\Xi(\theta)$ satisfy the condition (T, Ξ). Then we have*

$$\operatorname{Ord}_{\theta\downarrow 0} \min_{s\in\mathscr{R}_0(\Xi(\theta);p^{k_0})} |t_0+T(\theta)-s|$$

$$\times\, sub\, \sigma(P)(t_0+T(\theta), -a_1^{k_0}(t_0+T(\theta),\Xi(\theta))/2, \Xi(\theta))$$

$$\geq \operatorname{Ord}_{\theta\downarrow 0} h_{m-1}(t_0+T(\theta), -a_1^{k_0}(t_0+T(\theta),\Xi(\theta))/2, \Xi(\theta))^{1/2}.$$

5. Proof of Theorem 1.1

Let $n=2$, and let $(t_0,\xi^0)\in(0,\infty)\times S^1$. Let $a(t,\xi)$ and $b(t,\xi)$ be real analytic functions defined in a conic neighborhood $\mathscr{C}$ of (t_0,ξ^0). We assume that $a(t_0,\xi^0)=0$, $a(t,\xi)\geq 0$, $a(t,\xi)\not\equiv 0$ in $\mathscr{C}$ and $a(t,\xi)$ and $b(t,\xi)$ are positively homogeneous in ξ. Choose $\boldsymbol{e}\in S^1$, $\delta\equiv\delta(t_0,\xi^0)>0$ and $\theta_0\equiv\theta(t_0,\xi^0)>0$ so that $\boldsymbol{e}\perp\xi^0$, and

$$\{(t,\tilde{\Xi}_0(\theta));\ (t_0-\delta)_+\leq t\leq t_0+\delta \text{ and } |\theta|\leq\theta_0\}\subset\mathscr{C},$$

where $\tilde{\Xi}_0(\theta)\equiv\tilde{\Xi}_0(\theta;\xi_0,\boldsymbol{e})=(\xi^0+\theta\boldsymbol{e})/|\xi^0+\theta\boldsymbol{e}|$. We write

$$a^0(t,\theta)=a(t,\tilde{\Xi}_0(\theta)),\quad b^0(t,\theta)=b(t,\tilde{\Xi}_0(\theta)).$$

Then we have

$$a^0(t,\theta)=\sum_{k=l_0}^{\infty}a_k^0(t)\theta^k,\quad a_{l_0}^0(t)\not\equiv 0,$$

where $l_0\in\mathbf{Z}_+$. By the Weierstrass preparation theorem there are $r_0\in\mathbf{Z}_+$, a real analytic function $c^0(t,\theta)$ defined in $[0,\theta_0]$, real-valued continous functions $\tau_k^0(\theta)$ and $\sigma_k^0(\theta)$ $(1\leq k\leq r_0)$ defined in $[0,\theta_0]$ such that $c^0(t,\theta)>0$, $\tau_k^0(0)=\sigma_k^0(0)=0$ $(1\leq k\leq r_0)$, the $\tau_k^0(\theta)$ and $\sigma_k^0(\theta)$ can be expanded into convergent Puiseux series in $[0,\theta_0]$,

$$\tau_1^0(\theta)\leq\tau_2^0(\theta)\leq\cdots\leq\tau_{r_0}^0(\theta),\quad \sigma_k^0(\theta)\geq 0\quad(1\leq k\leq r_0)$$

$$a^0(t,\theta)=\theta^{l_0}c^0(t,\theta)\prod_{k=1}^{r_0}\{(t-t_0-\tau_k^0(\theta))^2+\sigma_k^0(\theta)\}\quad(\theta\in[0,\theta_0]),$$

with a modification of θ_0 if necessary, where $a^0(t,\theta)=\theta^{l_0}c^0(t,\theta)$ if $r_0=0$. Define

$$d^0(t,\theta)=\begin{cases}\theta^{l_0}\displaystyle\sum_{k=1}^{r_0}\prod_{l\neq k}\{(t-t_0-\tau_l^0(\theta))^2+\sigma_l^0(\theta)\} & \text{if } r_0>1,\\ \theta^{l_0} & \text{if } r_0\leq 1,\end{cases}\tag{5.1}$$

$$\mathscr{R}(\theta;a^0)=\begin{cases}\{t_0+\tau_k^0(\theta)+i\sqrt{\sigma_k^0(\theta)};\ 1\leq k\leq r_0\} & \text{if } r_0\geq 1,\\ \varnothing & \text{if } r_0=0.\end{cases}$$

Lemma 5.1. *There is $C > 0$ such that*

$$C^{-1} \min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda| \sqrt{d^0(t, \theta)} \le \sqrt{a^0(t, \theta)} \le C \min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda| \sqrt{d^0(t, \theta)}$$

for $t \in [(t_0 - \delta)_+, t_0 + \delta]$ and $\theta \in [0, \theta_0]$, with modifications of δ and θ_0 if necessary, where $\min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda| = 1$ if $r_0 = 0$.

Proof. When $r_0 = 0$, the lemma is trivial. Assume that $r_0 \ge 1$, and fix $(t, \theta) \in [(t_0 - \delta)_+, t_0 + \delta] \times [0, \theta_0]$. We choose $\nu_0 \in \mathbf{N}$ so that $1 \le \nu_0 \le r_0$ and

$$\min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda|^2 = (t - t_0 - \tau^0_{\nu_0}(\theta))^2 + \sigma^0_{\nu_0}(\theta).$$

Then we have

$$\begin{aligned} \min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda|^2 d^0(t, \theta) &= ((t - t_0 - \tau^0_{\nu_0}(\theta))^2 + \sigma^0_{\nu_0}(\theta)) d^0(t, \theta) \\ &\le r_0 a^0(t, \theta) \le r_0 \min_{\lambda \in \mathscr{R}(\theta; a^0)} |t - \lambda|^2 d^0(t, \theta). \qquad \square \end{aligned}$$

Let $1 \le k \le r_0$. Suppose that $b^0(t, \theta) \not\equiv 0$ in (t, θ). We note that $t_0 + \tau^0_k(\theta) \ge 0$ if $0 < \theta \ll 1$, since $t_0 > 0$. Applying the same argument as in §2 of [8], we can write

$$b^0(t_0 + \tau^0_k(\theta) + t, \theta) \sim \sum_{l=0}^{\infty} \beta_{k,l}(t) \theta^{\nu_k + l/L}, \quad \beta_{k,0}(t) \not\equiv 0,$$

where $L \in \mathbf{N}$ and $\nu_k \in \mathbf{Q}$. We define the Newton polygon $\Gamma^h_{b^0,k}$ of $t^h b^0(t_0 + \tau^0_k(\theta) + t, \theta)$ for $h = 0, 1, 2$ by

$$\Gamma^h_{b^0,k} = ch\left[\bigcup_{l \ge 0, \mu_{k,l} < \infty} (\{(\nu_k + l/L, h + \mu_{k.l})\} + [0, \infty)^2)\right],$$

where

$$\mu_{k,l} = \mathrm{Ord}_{t \downarrow 0}\, \beta_{k,l}(t)$$

and $ch[A]$ denotes the convex hull of A. If $b^0(t, \theta) \equiv 0$ in (t, θ), we define $\Gamma^h_{b^0,k} = \varnothing$ (see, also, §2 and §5 of [8]). We also denote by $\Gamma_{a^0,k}$ the Newton polygon of $a^0(t_0 + \tau^0_k(\theta) + t, \theta)$.

Lemma 5.2 (Lemma 2.2 of [8]). *Fix $h \in \{0, 1, 2\}$. The following two conditions* (i) *and* (ii) *are equivalent*:

(i) *If* $T(\theta)$ *is a real valued continuous function defined in* $[0,\theta_0]$, $T(\theta)\in C^\infty((0,\theta_0])$, $T(0)=0$, $t_0+T(\theta)>0$ *for* $\theta\in(0,\theta_0]$ *and* $T(\theta)$ *can be expanded into a formal Puiseux series, then*

$$\mathrm{Ord}_{\theta\downarrow 0}\left\{\min_{1\le k\le r_0}|T(\theta)-\tau_k^0(\theta)|^h|b^0(t_0+T(\theta),\theta)|\right\}\ge \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a^0(t_0+T(\theta),\theta)}.$$

(ii) $2\Gamma^h_{b^0,k}\subset\Gamma_{a^0,k}$ $(1\le k\le r_0)$ (see, also, Lemma 3.3 of [10]).

LEMMA 5.3. *Fix* $h\in\{0,1,2\}$. *Assume that*

$$2\Gamma^h_{b^0,k}\subset\Gamma_{a^0,k}\quad(1\le k\le r_0).$$

Then there is $C>0$ *such that*

$$\min_{1\le k\le r_0}|t-(t_0+\tau_k^0(\theta))|^h|b^0(t,\theta)|$$
$$\le C\sqrt{a^0(t,\theta)}\quad \textit{for}\ t\in[(t_0-\delta)_+,t_0+\delta]\ \textit{and}\ \theta\in[0,\theta_0],$$

with modifications of δ *and* θ_0 *if necessary.*

We proved Lemma 5.3 with $h=1$ in §5 of [8]. Lemma 5.3 with $h=0,2$ can be proved by the same arguments as in §5 of [8].

We assume that the Cauchy problem (CP) is C^∞ well-posed and has finite propagation property. We factorize $p(t,\tau,\xi)$ as (2.1):

$$p(t,\tau,\xi)=\prod_{k=1}^{r(j)}p^{j,k}(t,\tau,\xi)\quad\text{for}\ (t,\tau,\xi)\in[0,\delta_1]\times\mathbf{R}\times(\bar\Gamma_j\cap S^{n-1})$$

$(1\le j\le N_0)$. Fix j so that $1\le j\le N_0$. Assume that $1\le k_0\le r(j)$ and $m(j,k_0)=3$. Until the end of this section we omit the subscript j and the superscript j in the same manner as in §2. Now assume that $\hat a_2^{k_0}(t,\xi)\not\equiv 0$ in (t,ξ). It follows from (2.13), (3.3), Corollary 3.2 and Lemmas 5.2 and 5.3 that (2.8) and (2.19) of [11] with $\mathscr{R}(\xi)$ replaced by $\mathscr{R}_0(\xi)$ hold for $k=k_0$.

LEMMA 5.4. *Let* $(t_0,\xi^0)\in(0,\delta_1/2)\times(\Gamma\cap S^{n-1})$, *and let* $T(\theta)\in C^\infty((0,\theta_0])\cap C([0,\theta_0])$ *be a real-valued function satisfying the following*:

$T(0)=0$, $t_0+T(\theta)>0$ *for* $\theta\in[0,\theta_0]$ *and*

$T(\theta)$ *can be expanded into a formal Puiseux series.*

Then we have

$$\text{(5.2)}\qquad \mathrm{Ord}_{\theta\downarrow 0}\Bigg\{\min_{s\in\mathscr{R}_0(\tilde{\Xi}_0;p^{k_0})}|t_0+T(\theta)-s| \times \mathit{sub}\,\sigma(P^{k_0})(t_0+T(\theta),(\hat{a}_3^{k_0}(\cdot)/2)^{1/3}-a_1^{k_0}(\cdot)/3,\tilde{\Xi}_0(\theta))\hat{a}_2^{k_0}(\cdot)\Bigg\} \ge \mathrm{Ord}_{\theta\downarrow 0}(D_3^{k_0}(\cdot)\hat{a}_2^{k_0}(\cdot))^{1/2},$$

where $(\cdot)=(t_0+T(\theta),\tilde{\Xi}_0(\theta))$.

Proof. (2.44) yields

$$(\hat{a}_2^{k_0}(t,\xi)/3)h_2(t,(\hat{a}_3^{k_0}(t,\xi)/2)^{1/3},\xi;\hat{p}^{k_0})^{1/2}\le(D_3^{k_0}(t,\xi)(\hat{a}_2^{k_0}(t,\xi)/3))^{1/2}.$$

This, with Corollary 3.2, yields

$$\mathrm{Ord}_{\theta\downarrow 0}\Bigg\{\min_{s\in\mathscr{R}_0(\tilde{\Xi}_0;p^{k_0})}|t_0+T(\theta)-s| \times \mathit{sub}\,\sigma(P^{k_0})(t_0+T(\theta),A^{k_0}(\cdot)-a_1^{k_0}(\cdot)/3,\tilde{\Xi}_0(\theta))\hat{a}_2^{k_0}(\cdot)\Bigg\} \ge \mathrm{Ord}_{\theta\downarrow 0}(D_3^{k_0}(\cdot)\hat{a}_2^{k_0}(\cdot))^{1/2},$$

where $(\cdot)=(t_0+T(\theta),\tilde{\Xi}_0(\theta))$. Therefore, the lemma easily follows from (2.13) and (2.33)–(2.35). □

We may assume that $D_3^{k_0}(t,\xi)\not\equiv 0$ in (t,ξ). Indeed, if $D_3^{k_0}(t,\xi)\equiv 0$ in (t,ξ), then Lemma 5.4 implies that $\mathit{sub}\,\sigma(P^{k_0})(t,(\hat{a}_3^{k_0}(t,\xi)/2)^{1/3}-a_1^{k_0}(t,\xi)/3,\xi)\equiv 0$ in (t,ξ) and, therefore, (2.7) holds. Taking $a^0(t,\theta)=D_3^{k_0}(t,\tilde{\Xi}_0(\theta))\hat{a}_2^{k_0}(t,\tilde{\Xi}_0(\theta))$ in Lemma 5.1, Lemma 5.1 implies that (2.7) are valid if and only if

$$|(\hat{a}_2^{k_0}(\cdot)\,\mathit{sub}\,\sigma(P^{k_0}(t,(\hat{a}_3^{k_0}(\cdot)/2)^{1/3}-a_1^{k_0}(\cdot)/3,\tilde{\Xi}_0(\theta))^2|\le d^0(t,\theta)$$

for each $(t_0,\xi^0)\in[0,\delta_1]\times S^{n-1}$ with $D_3^{k_0}(t_0,\xi^0)=0$ and $(t,\theta)\in[(t_0-\delta)_+,t_0+\delta]\times[-\theta_0,\theta_0]$, where $(\cdot)=(t,\tilde{\Xi}_0(\theta))$ and $d^0(t,\theta)$ is defined by (5.1), since $\mathscr{R}(\theta;a^0)=\mathscr{R}_0(\tilde{\Xi}_0(\theta);p^{k_0})$. Choose $L\in\mathbf{N}$ so that $\tau_l^0(s^L)+i\sqrt{\sigma_l^0(s^L)}$ $(1\le l\le r_0)$ are real analytic in a neighborhood of $s=0$. We put

$$\tilde{d}(t,s)=d^0(t,s^L),\quad \tilde{a}(t,s)=\hat{a}_3^{k_0}(t,\tilde{\Xi}_0(s^L))/2$$

which are real analytic in (t,s). Moreover, $\tilde{d}(t,s)$ is a polynomial of t. Note that $\tilde{d}(t,s)$ depends on (t_0,ξ^0). It follows from Hironaka's resolution theorem that for each $(t_0,\xi^0) \in (0,\delta_1/2] \times S^{n-1}$ with $D_3^{k_0}(t_0,\xi^0) = 0$ there are an open neighborhood $U(t_0)$ of $(t,s) = (t_0,0)$ in $(0,\delta_1] \times \mathbf{R}$, a real analytic manifold $\tilde{U}(t_0)$, a proper analytic mapping $\varphi \equiv \varphi(t_0) : \tilde{U}(t_0) \ni \tilde{u} \mapsto \varphi(\tilde{u})$ $(\equiv \varphi(\tilde{u};t_0)) \in U(t_0)$ satisfying the following:

(i) $\varphi : \tilde{U}(t_0) \backslash \tilde{A} \to U(t_0) \backslash A$ is an isomorphism, where $A = \{(t,s) \in U(t_0);\ \tilde{a}(t,s) = 0\}$ and $\tilde{A} = \varphi^{-1}(A)$.

(ii) For each $p \in \tilde{U}(t_0)$ there are local analytic coordinates X $(\equiv X^p) = (X_1,X_2)$ $(= (X_1^p,X_2^p))$ centered at p, $\kappa_1,\kappa_2 \in \mathbf{Z}_+$, a neighborhood $\tilde{U}(t_0;p)$ of p and a real analytic function $e(X)$ in $\tilde{V}(t_0;p)$ such that $e(X) \neq 0$ in $\tilde{V}(t_0;p)$ and

$$\tilde{a}(\varphi(\tilde{u})) = e(X(\tilde{u}))X_1(\tilde{u})^{\kappa_1}X_2(\tilde{u})^{\kappa_2} \quad (\tilde{u} \in \tilde{U}(t_0;p)),$$

where $\tilde{V}(t_0;p) = \{X(\tilde{u});\ \tilde{u} \in \tilde{U}(t_0;p)\}$ (see [1]).

Define $\tilde{\varphi}$ $(\equiv \tilde{\varphi}(t_0;p)) : \tilde{V}(t_0;p) \to U(t_0)$ by $\tilde{\varphi}(X(\tilde{u}))$ $(\equiv \tilde{\varphi}(X^p(\tilde{u});t_0,p)) = \varphi(\tilde{u})$ $(\equiv \varphi(\tilde{u};t_0))$ for $\tilde{u} \in \tilde{U}(t_0;p)$. Then we have

$$\tilde{a}(\tilde{\varphi}(X)) = e(X)X_1^{\kappa_1}X_2^{\kappa_2} \quad (X \in \tilde{V}(t_0;p)).$$

Putting $X_l = \tilde{X}_l^3$ $(l = 1,2)$, we have

$$\tilde{a}(\varphi^0(\tilde{X}))^{1/3} = e(\tilde{X}_1^3,\tilde{X}_2^3)^{1/3}\tilde{X}_1^{\kappa_1}\tilde{X}_2^{\kappa_2} \quad (\tilde{X} \in V^0(t_0;p)),$$

where $\varphi^0(\tilde{X}) = \tilde{\varphi}(\tilde{X}_1^3,\tilde{X}_2^3)$ and $V^0(t_0;p) = \{\tilde{X} = (\tilde{X}_1,\tilde{X}_2);\ (\tilde{X}_1^3,\tilde{X}_2^3) \in \tilde{V}(t_0;p)\}$. Put $U(t_0;p) = \{\varphi(\tilde{u});\ \tilde{u} \in \tilde{U}(t_0;p)\}$, $\tilde{a}^0(\tilde{X}) = \tilde{a}(\varphi^0(\tilde{X}))^{1/3}$ and $\varphi^0(\tilde{X}) = (t(\tilde{X}), s(\tilde{X}))$. Then

$$\begin{aligned}(5.3)\qquad &\min\left\{\min_{v \in \mathscr{R}_0(\tilde{\Xi}_0(s^L);p^{k_0})} |t-v|, 1\right\} \\ &\quad \times |sub\ \sigma(P^{k_0})(t,(\hat{a}_3^{k_0}(\cdot)/2)^{1/3} - a_1^{k_0}(\cdot)/3, \tilde{\Xi}_0(s^L))| \\ &\quad \le Ch_2(t,(\hat{a}_3^{k_0}(\cdot)/2)^{1/3}, \tilde{\Xi}_0(s^L);\hat{p}^{k_0})^{1/2} \quad \text{for } (t,s) \in U(t_0;p)\end{aligned}$$

if and only if

$$(5.4)\qquad |B(\tilde{X})^2| \le C\tilde{d}(t(\tilde{X}),s(\tilde{X})) \quad \text{for } \tilde{X} \in V^0(t_0;p),$$

where $(\cdot) = (t,\tilde{\Xi}_0(s^L))$ and

$$B(\tilde{X}) = \hat{a}_2^{k_0}(\cdots)\ sub\ \sigma(P^{k_0})(t(\tilde{X}), \tilde{a}^0(\tilde{X}) - a_1^{k_0}(\cdots), \tilde{\Xi}_0(s(\tilde{X})^L)),$$

$$(\cdots) = (t(\tilde{X}), \tilde{\Xi}_0(s(\tilde{X})^L)).$$

Note that $\tilde{d}(t(\tilde{X}), s(\tilde{X}))$ and $B(\tilde{X})$ are real analytic in $V^0(t_0; p)$. Let $\tilde{X}(\theta)$ be real analytic near $\theta = 0$. Then it follows from (5.2) that

$$\mathrm{Ord}_{\theta \downarrow 0}\, \tilde{d}(t(\tilde{X}(\theta)), s(\tilde{X}(\theta)))/2 \le \mathrm{Ord}_{\theta \downarrow 0}\, B(\tilde{X}(\theta)).$$

Lemmas 5.2 and 5.3 with $b^0(t_0 + t, \theta) = B(\tilde{X})$, $a^0(t_0 + t, \theta) = \tilde{d}(t(\tilde{X}), s(\tilde{X}))$, $(t, \theta) = \tilde{X}$ and $h = 0$ yield (5.4) and, then, (5.3). Let I be a compact sub-interval of $(0, \delta_1/2]$. Applying compactness argument, we can prove that (2.7) holds with $[0, \delta_1]$ and $\mathscr{R}(\xi)$ replaced by I and $\mathscr{R}_0(\xi; p^{k_0})$, respectively. Therefore, Lemma 2.2 shows that (2.6) holds with $[0, \delta_1]$ and $\mathscr{R}(\xi)$ replaced by I and $\mathscr{R}_0(\xi)$, respectively. Next assume that $\hat{a}_2^{k_0}(t, \xi) \equiv 0$ in (t, ξ). Then (2.13) and (2.14) yield $D_3^{k_0}(t, \xi)$ $(\equiv \hat{D}^{k_0}(t, \xi)) \equiv 0$ in (t, ξ). Similarly, (2.6), and (2.19) of [11] hold with $\mathscr{R}(\xi)$ replaced by $\mathscr{R}_0(\xi)$. Let $1 \le j \le N_0$ and $1 \le k_0 \le r(j)$ satisfy $m(j, k_0) = 2$. Applying Corollary 4.2 and the same argument as before, we can prove that (2.36) with $\mathscr{R}(\xi)$ replaced by $\mathscr{R}_0(\xi)$ holds. Since (2.19) of [11] holds with $\mathscr{R}(\xi)$ and $[0, \delta_1]$ replaced by $\mathscr{R}_0(\xi)$ and I, respectively, as proved above, Lemma 2.5 of [11] proves Theorem 1.1 with $I \subset (0, \delta_1/2]$. The interval $(0, \delta_1/2]$ is determined by the factorization (2.1). So, finally one can prove Theorem 1.1 (with any compact interval $I \subset (0, \infty)$).

6. Proof of Theorem 1.2

Assume that the hypotheses of Theorem 1.2 are fulfilled. Let $1 \le j \le N_0$, and let $(t_0, \xi^0) \in [0, \delta_1/2] \times (\Gamma_j \cap S^{n-1})$. We fix $l = 1$ or 2. Let $h(t, \xi)$ be defined in a semi-algebraic set U in $\mathbf{R}^{n+1}$. Then we say that $h(t, \xi)$ is a semi-algebraic function if the graph of $h(t, \xi)$ is a semi-algebraic set. Let $a(t, \xi)$ and $b(t, \xi)$ be semi-algebraic functions defined in a conic neighborhood of (t_0, ξ^0). We assume that $a(t, \xi)$ and $b(t, \xi)$ are positively homogeneous in ξ, $a(t, \xi) \ge 0$ and $a(t_0, \xi^0) = 0$. Choose $\hat{\delta} > 0$ so that

$$D_{\hat{\delta}} \equiv \{(t, \xi);\ |t - t_0|^2 + |\xi - \xi^0|^2 \le \hat{\delta}^2,\ |\xi| = 1 \text{ and } t \ge 0\} \subset [0, \delta_1] \times \Gamma_j.$$

We may assume that $a(t, \xi)$ and $b(t, \xi)$ are defined in $D_{\hat{\delta}}$. Then we say that the condition $(A\text{-}B)_l$ is satisfied if

$(A\text{-}B)_l$ there are $\delta \in (0, \hat{\delta}]$ and $C > 0$ satisfying

$$\min\left\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^l, 1\right\} |b(t, \xi)| \le C\sqrt{a(t, \xi)} \quad \text{for } (t, \xi) \in D_\delta.$$

LEMMA 6.1. *Assume that the condition $(A\text{-}B)_l$ is not satisfied. Then there are $\theta_0 > 0$, $T_l(\theta), \Xi_k^l(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ $(1 \le k \le n)$ such that $T_l(\theta)$ and $\Xi^l(\theta)$ $(\equiv (\Xi_1^l(\theta),\dots,\Xi_n^l(\theta)))$ satisfy the condition (T,Ξ) and*

$$\text{(6.1)} \qquad \mathrm{Ord}_{\theta\downarrow 0} \min\left\{\min_{s \in \mathscr{R}_0(\Xi^l(\theta))} |t_0 + T_l(\theta) - s|^l, 1\right\}|b(\cdot)| < (\mathrm{Ord}_{\theta\downarrow 0}\, a(\cdot))/2,$$

where $(\cdot) = (t_0 + T_l(\theta), \Xi^l(\theta))$.

PROOF. Let $\delta \in (0,\hat{\delta}]$. Define

$$A = \{(t,\xi,y) \in D_\delta \times \mathbf{R};\ y = a(t,\xi)\},$$

$$B = \{(t,\xi,y) \in D_\delta \times \mathbf{R};\ y = |b(t,\xi)|^2\},$$

$$C_l = \left\{(t,\xi,y) \in D_\delta \times \mathbf{R};\ y = \min\left\{\min_{s \in \mathscr{R}_0(\xi)} |t-s|^{2l}, 1\right\}\right\}.$$

It is obvious that A and B are semi-algebraic sets. Put

$$\Xi_0 = \{\xi \in S^{n-1};\ |\xi - \xi^0| \le \delta \text{ and } D_M(s_0,\xi) \ne 0 \text{ for some } s_0 \in [0,\infty)\},$$

$$\Xi_k = \{\xi \in S^{n-1};\ |\xi - \xi^0| \le \delta,\ D_{M-k+1}(s,\xi) = 0 \text{ for any } s \in [0,\infty) \text{ and}$$
$$D_{M-k}(s_0,\xi) \ne 0 \text{ for some } s_0 \in [0,\infty)\} \quad (1 \le k \le M).$$

Since the $D_k(t,\xi)$ are semi-algebraic, the Ξ_k are semi-algebraic set, $\Xi_\mu \cap \Xi_\nu = \varnothing$ if $\mu \ne \nu$, and

$$\bigcup_{k=0}^{M} \Xi_k = \{\xi \in S^{n-1};\ |\xi - \xi^0| \le \delta\}.$$

Choose $\delta' \in (0,1]$ so that

$$\{t + i\tau \in \mathbf{C};\ t \in [-\delta', t_0 + 2],\ \tau \in \mathbf{R} \text{ and } |\tau| \le \delta'\} \subset \Omega,$$

where Ω is the complex neighborhood of $[0,\infty)$ as appears in §1. We define

$$\mathscr{D}_k = \{(t,\xi) \in \mathbf{R} \times S^{n-1};\ \xi \in \Xi_k,\ D_{M-k}(t_1 + i\tau,\xi) = 0,\ t_1 \in [-\delta', t_0 + 2],$$
$$\tau \in [-\delta',\delta'],\ t_2 \ge 0,\ t_2^2 = t_1^2 \text{ and } t = (t_1 + t_2)/2\} \quad (0 \le k \le M),$$

$$\mathscr{D} = \bigcup_{k=0}^{M} \mathscr{D}_k.$$

Note that $\mathscr{D}_M = \emptyset$. Then we have

$$C_l = \{(t, \xi, y) \in D_\delta;\ \text{“}(\hat{s}, \xi) \in \mathscr{D} \text{ or } \hat{s} = t - 1\text{”}, \\ |t - s|^2 \geq |t - \hat{s}|^2 \text{ for any } (s, \xi) \in \mathscr{D} \text{ and } y = |t - \hat{s}|^{2l}\}.$$

This shows that C_l is a semi-algebraic set. Put

$$\Lambda_l = \{(\rho, t, \xi, \lambda) \in \mathbf{R}^{n+3};\ \text{there are } y, u, v, w \in \mathbf{R} \text{ satisfying} \\ (t, \xi, y) \in A,\ (t, \xi, u) \in B,\ (t, \xi, v) \in C_l,\ \rho y = 1, \\ w((|t - t_0|^2 + |\xi - \xi^0|^2)\rho u v + 1) = 1 \text{ and } \lambda = \rho u v w\}.$$

Then Λ_l is semi-algebraic and

$$\Lambda_l = \Bigg\{(\rho, t, \xi, \lambda) \in \mathbf{R} \times D_\delta \times \mathbf{R};\ \rho a(t, \xi) = 1, \text{ and} \\ \lambda = \rho \min\Big\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^{2l}, 1\Big\} |b(t, \xi)|^2 \\ \times \Big((|t - t_0|^2 + |\xi - \xi^0|^2)\rho \min\Big\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^{2l}, 1\Big\} |b(t, \xi)|^2 + 1\Big)^{-1}\Bigg\}.$$

For $\rho > 0$ we define

$$K(\rho) = \{(t, \xi) \in D_\delta;\ \rho a(t, \xi) = 1\}.$$

Then $K(\rho)$ is compact and there is $\rho_0 > 0$ such that $K(\rho) \neq \emptyset$ for $\rho \geq \rho_0$. Indeed, we can take

$$\rho_0^{-1} = \max\{a(t, \xi);\ (t, \xi) \in D_\delta\},$$

since $a(t_0, \xi^0) = 0$. This yields

$$\{\rho \in \mathbf{R};\ (\rho, t, \xi, \lambda) \in \Lambda_l \text{ for some } (t, \xi, \lambda) \in \mathbf{R}^{n+2}\} \supset \{\rho;\ \rho \geq \rho_0\}.$$

Therefore, we can define

$$f_l(\rho) = \sup\{\lambda;\ (\rho, t, \xi, \lambda) \in \Lambda_l \text{ for some } (t, \xi) \in \mathbf{R}^{n+1}\} \quad \text{for } \rho \geq \rho_0. \tag{6.2}$$

Note that

$$f_l(\rho) = \max\Bigg\{\frac{\rho \min\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^{2l}, 1\} |b(t, \xi)|^2}{((|t - t_0|^2 + |\xi - \xi^0|^2)\rho \min\{\min_{s \in \mathscr{R}_0(\xi)} |t - s|^{2l}, 1\} |b(t, \xi)|^2 + 1)}; \\ (t, \xi) \in K(\rho)\Bigg\},$$

since $K(\rho)$ is compact. It follows from Theorem A.2.8 of [3] that there are continuous functions $\tilde{T}_l(\rho)$, $\tilde{\Xi}^l(\rho)$ and $\lambda_l(\rho)$ such that $\tilde{T}_l(\rho)$, $\tilde{\Xi}^l(\rho)$ and $\lambda_l(\rho)$ can be expanded into convergent Puiseux series for $\rho \gg 1$ and

$$(6.3) \qquad (\rho, t_0 + \tilde{T}_l(\rho), \tilde{\Xi}^l(\rho), \lambda_l(\rho)) \in \Lambda_l, \quad f_l(\rho) = \lambda_l(\rho) \ (\geq 0)$$

(see, also, [7]). Since the condition $(A\text{-}B)_l$ does not hold, there is $\{(t_k, \xi^k)\} \subset D_\delta$ satisfying $(t_k, \xi^k) \to (t_0, \xi^0)$ and

$$(6.4) \qquad \min\left\{\min_{s \in \mathscr{R}_0(\xi^k)} |t_k - s|^l, 1\right\} |b(t_k, \xi^k)| / a(t_k, \xi^k)^{1/2} \to \infty \quad \text{as } k \to \infty.$$

Put $\delta_k = (|t_k - t_0|^2 + |\xi^k - \xi^0|^2)^{1/2}$ and $\rho_k = a(t_k, \xi^k)^{-1}$. Then we have $\delta_k \to 0$ and $\rho_k \to \infty$ as $k \to \infty$. (6.3), together with (6.2) and (6.4), gives

$$\lambda_l(\rho_k) \geq \rho_k \min\left\{\min_{s \in \mathscr{R}_0(\xi^k)} |t_k - s|^{2l}, 1\right\} |b(t_k, \xi^k)|^2$$
$$\times \left(\delta_k^2 \rho_k \min\left\{\min_{s \in \mathscr{R}_0(\xi^k)} |t_k - s|^{2l}, 1\right\} |b(t_k, \xi^k)|^2 + 1\right)^{-1} \to \infty \quad \text{as } k \to \infty,$$

since $\delta_k \to 0$ and $\rho_k \min\{\min_{s \in \mathscr{R}_0(\xi^k)} |t_k - s|^{2l}, 1\} |b(t_k, \xi^k)|^2 \to \infty$ as $k \to \infty$. So we have $\lambda_l(\rho) \to \infty$ as $\rho \to \infty$, which implies that

$$\min\left\{\min_{s \in \mathscr{R}_0(\tilde{\Xi}^l(\rho))} |t_0 + \tilde{T}_l(\rho) - s|^l, 1\right\}$$
$$\times |b(t_0 + \tilde{T}_l(\rho), \tilde{\Xi}^l(\rho))| a(t_0 + \tilde{T}_l(\rho), \tilde{\Xi}^l(\rho))^{-1/2} \to \infty,$$
$$(\tilde{T}_l(\rho), \tilde{\Xi}^l(\rho)) \to (0, \xi^0)$$

as $\rho \to \infty$. There is $L \in \mathbf{N}$ such that $\tilde{\Xi}^l(\rho^L)$ is real analytic in ρ $(\geq \rho_0^{1/L})$. We put $T_l(\theta) = \tilde{T}_l(\theta^{-L})$ and $\Xi^l(\theta) = \tilde{\Xi}^l(\theta^{-L})$. Here, if $t_0 + T_l(\theta) \equiv 0$, then we replace $T_l(\theta)$ by $T_l(\theta) + \theta^N$, where $N \gg 1$. We note that $\Xi_k^l(\theta)$ $(1 \leq k \leq n)$ are real analytic in $\theta \in [0, \theta_0]$, where $\theta_0 = \rho_0^{-1/L}$. Then we have (6.1). □

First we assume that $1 \leq k_0 \leq r(j)$ and $m(j, k_0) = 3$. We take

$$\begin{cases} a(t, \xi) = h_{m-1}(t, A^{j,k_0}(t, \xi) - a_1^{j,k_0}(t, \xi)/3, \xi), \\ b(t, \xi) = sub\, \sigma(P)(t, A^{j,k_0}(t, \xi) - a_1^{j,k_0}(t, \xi)/3, \xi) \end{cases}$$

and $l = 1$, where

$$A^{j,k_0}(t,\xi) = \nu^{j,k_0}(t,\xi)(\hat{a}_2^{j,k_0}(t,\xi)/3)^{1/2},$$

$$\nu^{j,k_0}(t,\xi) = \begin{cases} 1 & \text{if } \hat{a}_3^{j,k_0}(t,\xi) \geq 0, \\ -1 & \text{if } \hat{a}_3^{j,k_0}(t,\xi) < 0 \end{cases}$$

(see (2.11)). It is easy to see that the coefficients of the polynomial $p^{j,k_0}(t,\tau,\xi)$ of τ are semi-algebraic. It follows from Lemma 6.1 that there are $\theta_0 > 0$, $T(\theta), \Xi_k(\theta) \in C^\infty((0,\theta_0]) \cap C([0,\theta_0])$ $(1 \leq k \leq n)$ such that $T(\theta)$ and $\Xi(\theta)$ $(\equiv (\Xi_1(\theta), \dots, \Xi_n(\theta)))$ satisfy the condition (T, Ξ) and

$$\begin{aligned} \mathrm{Ord}_{\theta\downarrow 0} \Big\{ \min\Big\{ \min_{s \in \mathscr{R}_0(\Xi(\theta))} |t_0 + T(\theta) - s|, 1 \Big\} \\ \times \, sub\ \sigma(P)(t_0 + T(\theta), A^{j,k_0}(\cdot) - a_1^{j,k_0}(\cdot)/3, \Xi(\theta)) \Big\} \\ < \mathrm{Ord}_{\theta\downarrow 0}\, h_{m-1}(t_0 + T(\theta), A^{j,k_0}(\cdot) - a_1^{j,k_0}(\cdot)/3, \Xi(\theta))^{1/2} \end{aligned}$$

if the condition $(A\text{-}B)_1$ is not satisfied, where $(\cdot) = (t_0 + T(\theta), \Xi(\theta))$. Therefore, Lemma 3.6 implies that the condition $(A\text{-}B)_1$ is satisfied if the Cauchy problem (CP) is C^∞ well-posed and has finite propagation property. Next we take

$$\begin{cases} a(t,\xi) = \hat{a}_2^{j,k_0}(t,\xi), \\ b(t,\xi) = (\partial_\tau\, sub\ \sigma(P))(t, A^{j,k_0}(t,\xi) - a_1^{j,k_0}(t,\xi)/3, \xi) \end{cases} \tag{6.5}$$

and $l = 1$. Similarly, we can see that the condition $(A\text{-}B)_1$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (6.5) and (CP) is C^∞ well-posed and has finite propagation property. Let $z^0 = (t_0, \tau_0, \xi^0)$ satisfy $(\partial_\tau^\mu p)(z^0) = 0$ $(\mu = 0, 1, 2)$ and $p^{k_0}(z^0) = 0$. We take

$$\begin{cases} a(t,\xi) = h_{m-2}(t, -a_1(t,\xi;z^0)/3, \xi), \\ b(t,\xi) = Q(t, -a_1(t,\xi;z^0)/3, \xi; z^0) \end{cases} \tag{6.6}$$

and $l = 2$. It is easy to see that the coefficients of the polynomials $p(t,\tau,\xi;z^0)$ and $Q(t,\tau,\xi;z^0)$ of τ are semi-algebraic. Similarly, we can see that the condition $(A\text{-}B)_2$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (6.6) and (CP) is C^∞ well-posed and has finite propagation property. This implies that (L-2) for $[0,\delta_1/2]$ is satisfied if (L-1) for $[0,\delta_1/2]$ is satisfied. Now we assume that $1 \leq k_0 \leq r(j)$ and $m(j,k_0) = 2$. We take

$$\begin{cases} a(t,\xi) = h_{m-1}(t, -a_1^{j,k_0}(t,\xi)/2, \xi), \\ b(t,\xi) = sub\ \sigma(P)(t, -a_1^{j,k_0}(t,\xi)/2, \xi) \end{cases} \tag{6.7}$$

and $l = 1$. Repetition of the above argument and Lemma 4.3 shows that the condition $(A\text{-}B)_1$ is satisfied if $a(t,\xi)$ and $b(t,\xi)$ are given by (6.7) and (CP) is C^∞ well-posed and has finite propagation property. It follows from the above results and Lemma 2.3 of [11] that (2.9), (2.10) and (2.37) hold for $(t,\xi) \in [0,\delta_1/2] \times (\bar{\Gamma}_j \cap S^{n-1})$, since

$$h_2(t, A^{j,k}(t,\xi) - a_1^{j,k}(t,\xi)/3, \xi; p^{j,k}) \approx h_{m-1}(t, A^{j,k}(t,\xi) - a_1^{j,k}(t,\xi)/3, \xi)$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \cap S^{n-1})$ if $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 3$, and

$$2\hat{a}_2^{j,k}(t,\xi) \; (= h_1(t, -a_1^{j,k}(t,\xi)/2, \xi; p^{j,k})) \approx h_{m-1}(t, -a_1^{j,k}(t,\xi)/2, \xi)$$

for $(t,\xi) \in [0,\delta_1] \times (\bar{\Gamma}_j \cap S^{n-1})$ if $1 \le j \le N_0$, $1 \le k \le r(j)$ and $m(j,k) = 2$. Therefore, Lemma 2.5 of [11] implies that (L-1) for $[0,\delta_1/2]$ is satisfied, which proves Theorem 1.2 with $T = \delta_1/2$. The interval $[0,\delta_1/2]$ is determined by the factorization (2.1). So, finally one can prove Theorem 1.2 (with $I = [0,T]$ for any $T > 0$).

References

[1] Atiyah, M. F., Resolution of singularities and division of distributions, Comm. Pure Appl. Math. **23** (1970), 145–150.

[2] Bierstone, E., Extension of Whitney fields from subanalytic set, Invent. Math. **46** (1978), 277–300.

[3] Hörmander, L., The Analysis of Linear Partial Differential Operators II, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[4] Ivrii, V. Ja. and Petkov, V., Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Uspehi Mat. Nauk. **29** (1971), 3–70. (Russian: English translation in Russian Math. Surveys.)

[5] Kumano-go, H., Pseudo-Differential Operators, MIT Press, Cambridge, 1982.

[6] Seely, R. T., Extension of C^∞ functions defined in a half space, Proc. Amer. Math. Soc. **15** (1964), 625–626.

[7] Svensson, S. L., Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part, Ark. Mat. **8** (1969), 145–162.

[8] Wakabayashi, S., On the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable, J. Math. Soc. Japan **62** (2010), 95–133.

[9] Wakabayashi, S., On the Cauchy problem for a class of hyperbolic operators whose coefficients depend only on the time variable, Tsukuba J. Math. **39** (2015), 121–163.

[10] Wakabayashi, S., On the Cauchy problem for hyperbolic operators with double characteristics whose principal parts have time dependent coefficients, Funkcialaj Ekvacioj **63** (2020), 345–418.

[11] Wakabayashi, S., On the Cauchy problem for hyperbolic operators with triple charecteristics whose coefficients depend only on the time variable I, Tsukuba J. Math. **49** (2025), 1–50.

[12] Wakabayashi, S., An alternative proof of Ivrii-Petkov's necessary condition for C^∞ well-posedness of the Cauchy problem, located in https://wkbysh.sakura.ne.jp/.

[13] Wakabayashi, S., Remarks on semi-algebraic functions, located in https://wkbysh.sakura.ne.jp/.

[14] Wakabayashi, S., Remarks on semi-algebraic functions II, located in https://wkbysh.sakura.ne.jp/.

Seiichiro Wakabayashi
(professor emeritus)
University of Tsukuba
E-mail: wkbysh@wb3.so-net.ne.jp